The Arnold-Givental conjecture and moment Floer homology

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Abstract

In this article we prove the Arnold-Givental conjecture for a class of Lagrangian submanifolds in Marsden-Weinstein quotients which are fixpoint sets of some antisymplectic involution. For these Lagrangians the Floer homology cannot in general be defined by standard means due to the bubbling phenomenon. To overcome this difficulty we consider moment Floer homology whose boundary operator is defined by counting solutions of the symplectic vortex equations on the strip which have better compactness properties than the original Floer equations.

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1 Introduction

Assume that (M,ω) is a $2n$-dimensional compact symplectic manifold, $L \subset M$ is a compact Lagrangian submanifold of M, and $R \in \mathrm{Diff}(M)$ is an antisymplectic involution, i.e.

$$R^*\omega = -\omega, \quad R^2 = \mathrm{id}$$

whose fixpoint set is the Lagrangian

$$L = \mathrm{Fix}(R).$$

The Arnold-Givental conjecture gives a lower bound on the number of intersection points of L with a Hamiltonian isotopic Lagrangian submanifold which intersects L transversally in terms of the Betti numbers of L, more precisely

Conjecture (Arnold-Givental) *For $t \in [0,1]$ let $H_t \in C^\infty(M)$ be a smooth family of Hamiltonian function of M and denote by ϕ_H the one-time map of the flow of the Hamiltonian vector field X_{H_t} of H_t. Assume that L and $\phi_H(L)$ intersect transversally. Then the number of intersection points of L and $\phi_H(L)$ can be estimated from below by the sum of the $\mathbb{Z}_2$ Betti numbers of L, i.e.*

$$\#(L \cap \phi_H(L)) \geq \sum_{k=0}^{n} b_k(L;\mathbb{Z}_2).$$

Using the fact that the diagonal Δ is a Lagrangian submanifold of $(M \times M, \omega \oplus -\omega)$ which equals the fixpoint set of the antisymplectic involution of $M \times M$ given by interchanging the two factors, the Arnold conjecture with $\mathbb{Z}_2$ coefficients for arbitrary compact symplectic manifolds is a Corollary of the Arnold-Givental conjecture.

Corollary (Arnold conjecture) *Assume that for $t \in [0,1]$ there is a smooth family $H_t \in C^\infty(M)$ of Hamiltonian functions such that 1 is not an eigenvalue of $d\phi_H(x)$ for each fixpoint x of ϕ_H, then the number of fixpoints of ϕ_H can be estimated from below by the sum of the $\mathbb{Z}_2$ Betti numbers of M, i.e.*

$$\#\mathrm{Fix}(\phi_H) \geq \sum_{k=0}^{2n} b_k(M,\mathbb{Z}_2).$$

Up to now, the Arnold-Givental conjecture could only be proven under some additional assumptions. Givental proved it in [Gi] for $(\mathbb{C}P^n, \mathbb{R}P^n)$, Oh proved it in [Oh3] for real forms of compact Hermitian spaces with suitable conditions on the Maslov indices, Lazzarini proved it in [La] in the negative monotone case under suitable assumptions on the minimal Maslov number, and Fukaya, Oh, Ohta, Ono proved it in [FOOO] for semipositive Lagrangian submanifolds. In my thesis, see [Fr2], I introduced moment Floer homology and proved the Arnold-Givental conjecture for a class of Lagrangians in Marsden-Weinstein quotients

which satisfy some monotonicity condition. In this paper, this monotonicity condition will be removed.

1.1 Outline of the paper

In section 2 we give a heuristic argument why the Arnold-Givental conjecture should hold. This argument is based on Floer theory. We will ignore questions of bubbling and of transversality. The main point is that due to the antisymplectic involution some contribution to the boundary operator should cancel.

To describe the cancelation process one has to choose the almost complex structure in a non-generic way. For such a choice of almost complex structure one cannot in general expect to achieve transversality. To overcome this problem we consider in section 3 abstract perturbations as in [FOn] and prove some transversality result specific for problems of Arnold-Givental type. We will need that later to compute moment Floer homology. In my thesis, see [Fr2], this technique was not available to me and hence I could only compute moment Floer homology under some monotonicity assumption.

In section 4 we introduce moment Floer homology. The chain complex of moment Floer homology is generated by the intersection points of some Lagrangian submanifold of a Marsden-Weinstein quotient which is the fixpoint set of an antisymplectic involution with its image under a generic Hamiltonian isotopy. The boundary operator is defined by counting solutions of the symplectic vortex equations on the strip. These equations contain an equivariant version of Floer's equation and a condition which relates the curvature to the moment map. We prove under some topological restrictions on the envelopping manifold a compactness theorem which allows us to prove that the boundary operator is well-defined. To compute the moment Floer homology we set the Hamiltonian equal to zero. This is the infinite dimensional analogon of a Morse-Bott situation. Combining the techniques developed in section 3 together with the approach to Morse-Bott theory described in the appendix by defining a boundary operator by counting flow lines with cascades we prove that moment Floer homology is isomorphic to the singular homology of the Lagrangian in the Marsden-Weinstein quotient with coefficients in some Novikov ring.

In the appendix we develop some approach to define Morse-Bott homology by counting flow-lines with cascades. We prove that for a given Morse-Bott function and for generic choice of a Riemannian metric on the manifold and of a Morse-function and a Riemannian metric on the critical manifold transversality for flow-lines with cascades can be achieved and hence the boundary operator is well-defined. We show that Morse-Bott homology is independent of the Morse-Bott function and hence isomorphic to the ordinary Morse homology.

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2 Arnold-Givental conjecture

We will give in this section a heuristic argument based on Floer theory why the Arnold-Givental conjecture should hold. We refer the reader to Floer's original papers [Fl1, Fl2, Fl3, Fl4, Fl5] and to [S2] for an introduction into the topic of Floer theory. We will here just introduce the main objects of the theory. Moreover, we completely ignore questions of bubbling and of transversality. We will address the question of transversality in section 3. We hope that these techniques combined with the techniques developed in [FOn] and [FOOO] to overcome the bubbling phenomenon should lead to a proof of the Arnold-Givental conjecture in general. However, in this paper we will not persecute such an approach but consider instead in section 4 moment Floer homology where no bubbling occurs.

We abbreviate by Γ the group

$$\Gamma=\frac{\pi_2(M,L)}{\mathrm{ker}I_\mu\cap\mathrm{ker}I_\omega}$$

where $I_\omega\colon\pi_2(M,L)\to\mathbb{R}$ and $I_\mu\colon\pi_2(M,L)\to\mathbb{Z}$ are the homomorphisms induced from the symplectic structure ω respectively the Maslov number μ. We refer the reader to [RS1, RS2, SZ] for a discussion of the Maslov index. Following [HS] we denote the Novikov ring $\Lambda=\Lambda_\Gamma$ as the ring consisting of formal sums

$$r=\sum_{\gamma\in\Gamma}r_\gamma\gamma,\quad r_\gamma\in\mathbb{Z}_2$$

which satisfy the finiteness condition

$$\#\{\gamma\in\Gamma:r_\gamma\neq 0,\ I_\omega(\gamma)\geq\kappa\}<\infty,\quad\forall\,\kappa\in\mathbb{R}.$$

Note that since the coefficients are taken in the field $\mathbb{Z}_2$ the Novikov ring is actually a field. It is naturally graded by I_μ.

To define the Floer chain complex we consider pairs $(\bar{x},x)\in C^\infty((\Omega,\Omega\cap\mathbb{R}),(M,L))\times C^\infty(([0,1],\{0,1\}),(M,L))$ [1] for the half-disk $\Omega=\{z\in\mathbb{C}:|z|\leq 1,\ \mathrm{Im}(z)\geq 0\}$ which satisfy the following conditions

$$\dot{x}(t)=X_{H_t}(x(t)),\quad\bar{x}(e^{i\pi t})=x(t),\quad t\in[0,1].$$

We introduce an equivalence relation on these pairs by

$$(\bar{x},x)\cong(\bar{y},y)\Longleftrightarrow x=y,\ \omega(\bar{x})=\omega(\bar{y}),\ \mu(\bar{x})=\mu(\bar{y})$$

[1] We abbreviate for manifolds $A_2\subset A_1$ and $B_2\subset B_1$ by $C^\infty((A_1,A_2),(B_1,B_2))$ the space of smooth maps from A_1 to B_1 which map A_2 to B_2.

and denote the set of equivalence classes by $\mathscr{C}$, recalling the fact that this set may be interpreted as the critical set of an action functional on a covering of the space of paths in M which connect two points of the Lagrangian L. The Floer chain complex $CF_*(M,L;H)$ can now be defined as the graded $\mathbb{Z}_2$ vector space consisting of formal sums

$$\xi = \sum_{c\in\mathscr{C}} \xi_c c, \quad \xi_c \in \mathbb{Z}_2$$

satisfying the finiteness condition

$$\#\{c = [\bar{x}, x] \in \mathscr{C} : \xi_c \neq 0,\ I_\omega(\bar{x}) \geq \kappa\} < \infty, \quad \forall\, \kappa \in \mathbb{R}.$$

The grading of CF_* is induced from the Maslov index. The natural action of Γ on CF_* by cocatenation induces on CF_* the structure of a graded vector space over the graded field Λ.

To define the boundary operator we choose a smooth family of ω-compatible almost complex structures J_t for $t \in [0,1]$ and count for two critical points $[\bar{x}, x], [\bar{y}, y] \in \mathscr{C}$ solutions $u \in C^\infty([0,1]\times\mathbb{R}, M)$ of the following problem

$$\begin{aligned}
\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) &= 0 \qquad\qquad (1)\\
u(s,j) \in L, &\qquad j \in \{0,1\}\\
\lim_{s\to-\infty} u(s,t) &= x(t)\\
\lim_{s\to\infty} u(s,t) &= y(t)\\
\bar{x}\#[u]\#\bar{y} &= 0.
\end{aligned}$$

Here the limites are uniform in the t-variable with respect to the C^1-topology and $\#$ denotes cocatenation. For generic choice of the almost complex structures the moduli space $\tilde{\mathcal{M}}([\bar{x}, x], [\bar{y}, y])$ of solutions of (1) is a smooth manifold of dimension

$$\dim(\tilde{\mathcal{M}}([\bar{x}, x], [\bar{y}, y])) = \mu(\bar{x}) - \mu(\bar{y}).$$

If $[\bar{x}, x]$ is different from $[\bar{y}, y]$ the group $\mathbb{R}$ acts freely on the solutions of (1) by timeshift

$$u(s,t) \mapsto u(s+r,t), \quad r \in \mathbb{R}.$$

We denote the quotient by

$$\mathcal{M}([\bar{x}, x], [\bar{y}, y]) = \tilde{\mathcal{M}}([\bar{x}, x], [\bar{y}, y])/\mathbb{R}.$$

If we ignore the question of bubbling the manifold $\mathcal{M}([\bar{x}, x], [\bar{y}, y])$ for critical points $[\bar{x}, x], [\bar{y}, y] \in \mathscr{C}$ satisfying $\mu(\bar{x}) - \mu(\bar{y}) = 1$ is compact and we may define the $\mathbb{Z}_2$ numbers

$$n([\bar{x}, x], [\bar{y}, y]) := \#\mathcal{M}([\bar{x}, x], [\bar{y}, y]) \bmod 2.$$

The Floer boundary operator is now defined as the linear extension of

$$\partial[\bar{x}, x] = \sum_{\substack{[\bar{y},y]\in\mathscr{C},\\ \mu(\bar{y})=\mu(\bar{x})-1}} n([\bar{x}, x], [\bar{y}, y])[\bar{y}, y].$$

Ignoring again the bubbling problem one can "prove"

$$\partial^2 = 0$$

and hence one can define

$$HF_*(M, L; H, J) = \frac{\ker\partial}{\mathrm{Im}\partial}.$$

One can show - always ignoring the bubbling problem - that for different choices of H and J there are canonical isomorphisms between the graded Λ vector spaces and hence Floer homology $HF_*(M, L)$ is defined independent of H and J. It follows from its definition that the dimension of HF_* as Λ vector space gives a lower bound on the number of intersection points of L and $\phi_H(L)$.

To "compute" the Floer homology we consider the case where $H = 0$. Actually, in this case L and $\phi_H(L) = L$ do not intersect transversally but still cleanly [2]. The finite dimensional analogon of clean intersections in Floer theory are Morse-Bott functions. In our case the critical manifold consists of different copies of the Lagrangian L indexed by the group Γ. Following the approach developed in the appendix one can still define Floer homology in the case of clean intersections. To do that one chooses a Morse function on the critical manifold. The critical points of this Morse function give a basis for the chain complex. The boundary operator is defined by counting flow lines with cascades. In our case, a cascade is just a nonconstant solution $u \in C^\infty([0,1] \times \mathbb{R}, M)$ of the unperturbed Floer equation

$$\begin{aligned} \partial_s u + J_t(u)\partial_t u &= 0, \\ u(s,j) \in L, & \quad\quad j \in \{0,1\} \end{aligned}$$

which converges uniformly in the t-variable to points on L as s goes to $\pm\infty$. A flow line with zero cascades is just a Morse flow line. A flow line with one cascade consists of a piece of a Morse flow line a cascade which converges on the left to the endpoint of this piece and which converges on the right to the initial point of a second piece of a Morse flow line, see appendix A for details. The boundary operator now splits naturally

$$\partial = \partial^0 + \partial^1$$

where ∂^0 consists of the flow lines with zero cascades and ∂^1 consists of the flow lines with at most one cascade. Note that ∂^0 is precisely the Morse boundary operator. If one can show that ∂^1 is zero it follows that

$$HF_*(M, L) = HM_*(L; \mathbb{Z}_2) \times_{\mathbb{Z}_2} \Lambda$$

where $HM_*(L; \mathbb{Z}_2)$ is the Morse homology of L with $\mathbb{Z}_2$ coefficients, which is isomorphic to the singular homology of L with $\mathbb{Z}_2$ coefficients, see [Sch1].

[2] Two submanifolds $L_1, L_2 \subset M$ are said to intersect cleanly if their intersection is a manifold such that for each $x \in L_1 \cap L_2$ it holds that $T_x(L_1 \cap L_2) = T_xL_1 \cap T_xL_2$.

It now remains to explain why one has to expect that ∂^1 vanishes. Since the Lagrangian L is the fixpoint set of the antisymplectic involution R there is an induced involution on the space of cascades. More precisely, since we ignore questions of transversality we may now choose the family of ω compatible almost complex structures J_t independent of the t-variable and impose furthermore the condition that J is antiinvariant under R, i.e.

$$R^*J = -J.$$

When J is independent of the t-variable a cascade corresponds to a nonconstant J-holomorphic disk satisfying Lagrangian boundary conditions, i.e.

$$\begin{aligned} \partial_s u + J(u)\partial_t u &= 0 \\ u(s,j) \in L, \qquad & j \in \{0,1\}. \end{aligned} \tag{2}$$

The antisymplectic involution R induces a natural involution on the solutions of the problem above defined by

$$I_1 u(s,t) = R(u(s,1-t)).$$

This involution does not act freely in general, but it was observed by K.Ono that on the "lanterns", the fixpoints of the involution I_1, one can define a new involution defined by

$$I_2 u(s,t) = \begin{cases} u(s,t+1/2) & 0 \leq t \leq 1/2 \\ u(s,t-1/2) & 1/2 \leq t \leq 1 \end{cases}$$

Finally, on the "double lanterns", the fixpoints of the involution I_2, one can define a third involution given by

$$I_3 u(s,t) = \begin{cases} u(s,t+1/4) & 0 \leq t \leq 3/4 \\ u(s,t-3/4) & 3/4 \leq t \leq 1 \end{cases}$$

and so on. Since we consider only nonconstant solution, there exists $m \in \mathbb{N}$ such that I_m acts freely. Hence one may expect that there is an even number of flow lines with at most one cascade. Since we are working with $\mathbb{Z}_2$ coefficients this implies that ∂^1 is zero.

3 Transversality

For general symplectic manifolds one cannot expect to find an R-invariant ω-compatible almost complex structure which is regular so that the relevant moduli spaces have the structure of finite dimensional manifolds. Hence to make precise the heuristic argument at the end of the last section one has to use abstract perturbation. To do that one considers solutions of (2) modulo the natural $\mathbb{R}$ action as the zero set of a section of an infinite dimensional Banach manifold $\mathcal{B}$ into a Banach bundle $\mathcal{E}$ over it. There are natural extensions of

the involutions I_k to involutions of the Banach manifold, so that again I_1 is defined on the whole space, I_2 on the fixpoint set of I_1 and so on. The idea of our abstract perturbation is now to perturb our section in such a way that the perturbed section intersects the zero section transversally but that its zero set is still invariant under the involutions. There are two problems we have to overcome. The first one is that one cannot define invariants from the zero set of an arbitrary section from in infinite dimensional space and hence one should define the section in some "finite dimensional" neighbourhood of the zero set of the original section. This problem was solved in [FOn] by using Kuranishi structures. The second problem is that for each involution one should find extensions $I_k^{T\mathcal{B}}$ and $I_k^{\mathcal{E}}$ to the tangent space $T\mathcal{B}$ or the bundle $\mathcal{E}$ respectively restricted to the domain of definition of I_k in order to achieve that the zero set of the perturbed section is invariant under the involutions.

3.1 Banach spaces

We interpret solutions of (2) as the zero-set of a smooth section from a Banach manifold $\tilde{\mathcal{B}}$ to a Banachbundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{B}}$. To define $\tilde{\mathcal{B}}$ we first have to introduce some weighted Sobolev norms. Choose a smooth cutoff function $\beta \in C^\infty(\mathbb{R})$ such that $\beta(s) = 0$ for $s < 0$ and $\beta(s) = 1$ for $s > 1$. Choose $\delta > 0$ and define $\gamma_\delta \in C^\infty(\mathbb{R})$ by

$$\gamma_\delta(s) := e^{\delta\beta(s)s}.$$

Let Ω be an open subset of the strip $\mathbb{R} \times [0,1]$. For $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$ we define the $|| \, ||_{k,p,\delta}$-norm for $v \in W^{k,p}(\Omega)$ by

$$||v||_{k,p,\delta} := \sum_{i+j \leq k} ||\gamma_\delta \cdot \partial_s^i \partial_t^j v||_p.$$

We introduce the following weighted Sobolev spaces

$$W_\delta^{k,p}(\Omega) := \{v \in W^{k,p}(\Omega) : ||v||_{k,p,\delta} < \infty\} = \{v \in W^{k,p}(\Omega) : \gamma_\delta v \in W^{k,p}(\Omega)\}.$$

We abbreviate

$$L_\delta^p(\Omega) := W_\delta^{0,p}(\Omega).$$

Fix a real number $p > 2$ and a Riemannian metric g on TM. The Banach manifold $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_\delta^{1,p}(M,L)$ consists of $W_{loc}^{1,p}$-maps u from the strip $\mathbb{R} \times [0,1]$ to M which map the boundary of the strip to the Lagrangian L and satisfy in addition the following conditions.

B1: There exists a point $x^- \in L$, a real number T_1, and an element $v_1 \in W_\delta^{1,p}((-\infty,-T_1) \times [0,1], T_{x^-}M)$ such that

$$u(s,t) = \exp_{x^-}(v_1(s,t)), \quad s < -T_1.$$

Here the exponential map is taken with respect to the metric g.

B2: There exists a point $x^+ \in L$, a real number T_2, and an element $v_2 \in W^{1,p}_\delta((T_2,\infty)\times[0,1], T_{x^+}M)$ such that

$$u(s,t) = \exp_p(v_2(s,t)), \quad s > T_2.$$

We introduce the Banach bundle $\tilde{\mathcal{E}}$ over the Banach-manifold $\tilde{\mathcal{B}}$ whose fiber over $u \in \tilde{\mathcal{B}}$ is given by

$$\tilde{\mathcal{E}}_u := L^p_\delta(u^*TM).$$

Define the section $\tilde{\mathcal{F}} \colon \tilde{\mathcal{B}} \to \tilde{\mathcal{E}}$ by

$$\tilde{\mathcal{F}}(u) := \partial_s u + J(u)\partial_t u$$

for $u \in \tilde{\mathcal{B}}$. Note that the zero set $\tilde{\mathcal{F}}^{-1}(0)$ consists of solutions of (2). The vertical differential of $\tilde{\mathcal{F}}$ at $u \in \mathcal{F}^{-1}(0)$ is given by

$$\tilde{D}_u\xi := D\tilde{\mathcal{F}}(u)\xi = \partial_s\xi + J(u)\partial_t\xi + \nabla_\xi J(u)\partial_t u, \quad \xi \in T_u\tilde{\mathcal{B}},$$

where ∇ denotes the Levi-Civita connection of the metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. There is a natural action of the group $\mathbb{R}$ on $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{E}}$ given by timeshift

$$u(s,t) \mapsto u(s+r,t), \quad r \in \mathbb{R}.$$

We denote the quotient by

$$\mathcal{B} := \tilde{\mathcal{B}}/\mathbb{R}, \quad \mathcal{E} := \tilde{\mathcal{E}}/\mathbb{R},$$

by

$$\mathcal{F} \colon \mathcal{B} \to \mathcal{E}$$

we denote the section induced from $\tilde{\mathcal{F}}$ and by D_u we denote the vertical differential of $\mathcal{F}$ for $u \in \mathcal{F}^{-1}(0)$. By induction we define in the obvious way for every $k \in \mathbb{N}$ smooth involutions I_k on $\mathcal{B}_k = \mathrm{Fix}(I_{k-1})$ where we set $\mathcal{B}_1 = \mathcal{B}$. Our aim is to construct natural extensions of these involutions to $T\mathcal{B}$ and $\mathcal{E}$ restricted to their domain of definition. We prove the following theorem.

Theorem 3.1 *For each $k \in \mathbb{N}$ there exist smooth involutative bundle maps*

$$I_k^{T\mathcal{B}} \colon T\mathcal{B}|_{\mathcal{B}_k} \to T\mathcal{B}|_{\mathcal{B}_k}, \quad I_k^{\mathcal{E}} \colon \mathcal{E}|_{\mathcal{B}_k} \to \mathcal{E}|_{\mathcal{B}_k}$$

which extend I_k and which have the following properties.

(i) *The bundle maps commute on their common domain of definition, i.e.*

$$I_k^{T\mathcal{B}} \circ I_\ell^{T\mathcal{B}}\xi = I_\ell^{T\mathcal{B}} \circ I_k^{T\mathcal{B}}\xi, \quad I_k^{\mathcal{E}} \circ I_\ell^{\mathcal{E}}\eta = I_\ell^{\mathcal{E}} \circ I_k^{\mathcal{E}}\eta, \quad \xi \in T\mathcal{B}|_{\mathcal{B}_\ell},\ \eta \in \mathcal{E}|_{\mathcal{B}_\ell}$$

for $\ell \geq k$.

(ii) *For $k \in \mathbb{N}$ and $u \in \mathcal{F}^{-1}(0) \cap \mathcal{B}_k$ the vertical differential of $\mathcal{F}$ commutes with the involutions modulo a compact operator. More precisely, there exists a compact operator $Q_u \colon T_u\mathcal{B} \to \mathcal{E}_u$ such that*

$$I_k^{\mathcal{E}} \circ D_u - D_u \circ I_k^{T\mathcal{B}} = I_k^{\mathcal{E}} \circ Q_u - Q_u \circ I_k^{T\mathcal{B}}.$$

Moreover, $I_k^{\mathcal{E}} \circ Q_u - Q_u \circ I_k^{T\mathcal{B}}$ vanishes on $T_u\mathcal{B}_k$.

(iii) *The restriction of of $I_k^{T\mathcal{B}}$ to the tangent space of $\mathcal{B}_k$ equals the differential of I_k, i.e.*

$$I_k^{T\mathcal{B}}|_{T\mathcal{B}_k} = dI_k.$$

3.2 A sequence of involutions

The aim of this subsection is to prove Theorem 3.1. Since the involution we want to construct are independent of the s-variable it is most convenient to define them on the space of paths whose endpoints lie on the Lagrangian L. We define for a real number $p > 1$ the path space

$$\mathscr{P} := \mathscr{P}^{1,p}(M, L) := W^{1,p}(([0,1], \{0,1\}), (M, L)).$$

For $x \in \mathscr{P}$ the tangent space of $\mathscr{P}$ at x is given by

$$T_x\mathscr{P} = W^{1,p}(([0,1], \{0,1\}), (T_x^*M, T_x^*L)).$$

We define the bundle $\mathscr{E}$ over $\mathscr{P}$ by

$$\mathscr{E} = L^p([0,1], T_x^*M).$$

Note that $T\mathscr{P}$ is a subbundle of $\mathscr{E}$. As before we set

$$\mathscr{P}_1 := \mathscr{P}$$

and define the first involution $\mathscr{I}_1 \in \mathrm{Diff}(\mathscr{P}_1)$ by

$$\mathscr{I}_1 x(t) := R(x(1-t)), \quad x \in \mathscr{P}_1.$$

By induction on $k \in \mathbb{N}$ we define for $k \geq 2$

$$\mathscr{P}_k := \mathrm{Fix}(\mathscr{I}_{k-1}), \quad \mathscr{I}_k x(t) := x(t + \frac{1}{2^{k-1}} - \lfloor t + \frac{1}{2^{k-1}} \rfloor), \quad x \in \mathscr{P}_k.$$

Here $\lfloor\ \rfloor$ denote the Gauss brackets, i.e. the largest integer which is smaller then a given real number

$$\lfloor r \rfloor := \max\{n \in \mathbb{Z} : n \leq r\}, \quad r \in \mathbb{R}.$$

Observe that if the index of integrability p is greater than two and $u \in \tilde{\mathcal{B}}$ has the property that $u_s(t) := u(s,t) \in \mathscr{P}_k$ for every $s \in \mathbb{R}$ and some $k \in \mathbb{N}$, then $\mathscr{I}_k$ induces an involution on u which commutes with the $\mathbb{R}$ action on $\tilde{\mathscr{B}}$ so that the induced map in the quotient $\mathcal{B} = \tilde{\mathcal{B}}/\mathbb{R}$ coincides with the involution I_k.

We will find in this subsection an extension of these involutions to smooth involutative bundle maps $\mathscr{I}_k \colon \mathscr{E}|_{\mathscr{P}_k} \to \mathscr{E}|_{\mathscr{P}_k}$ such that the following properties are satisfied.

(i) The tangent bundle of the path space is invariant under the involutions

$$\mathscr{I}_k(T\mathscr{P}|_{\mathscr{P}_k}) = T\mathscr{P}|_{\mathscr{P}_k}.$$

Moreover,

$$\mathscr{I}_k|_{T\mathscr{P}_k} = d\mathscr{I}_k|_{\mathscr{P}_k}.$$

(ii) The involutions commute on their common domain of definition

$$\mathscr{I}_k \circ \mathscr{I}_\ell \xi = \mathscr{I}_\ell \circ \mathscr{I}_k \xi, \quad \xi \in \mathscr{E}|_{\mathscr{P}_\ell}$$

for $\ell \geq k$.

We will then define the involution $I_k^{T\mathcal{B}}$ and $I_k^{\mathcal{E}}$ of Theorem 3.1 as the induced maps of $\mathscr{I}_k$ on $T\mathcal{B}|_{\mathcal{B}_k}$ respectively $\mathcal{E}|_{\mathcal{B}_k}$. Property (i) of the involutions $\mathscr{I}_k$ guarantees that $I_k^{T\mathcal{B}}$ is well defined and guarantees assertion (iii) of Theorem 3.1. Assertion (i) of Theorem 3.1 follows from property (ii) of the involutions $\mathscr{I}_k$ and assertion (ii) of Theorem 3.1 will follow from our construction.

For $x \in \mathscr{P}$ the extension of the first involution to $\mathscr{E}$ is defined in the following way

$$\mathscr{I}_1\xi(t) := R^*\xi(1-t) \in \mathscr{E}_{\mathscr{I}_1 x}, \quad \xi \in \mathscr{E}_x.$$

If $x \in \mathscr{P}_2$, then $\mathscr{I}_1$ is a bounded linear involutative map of the Banach space $\mathscr{E}_x$ and leads to a decomposition

$$\mathscr{E}_x = \mathscr{E}_{x,-1} \oplus \mathscr{E}_{x,1},$$

where $\mathscr{E}_{x,-1}$ is the eigenspace of $\mathscr{I}_1|_{\mathscr{E}_x}$ to the eigenvalue -1 and $\mathscr{E}_{x,1}$ is the eigenspace to the eigenvalue 1. The two projections to the eigenspaces are given by

$$\pi_{x,1} = \frac{1}{2}\Big(\mathrm{id}|_{\mathscr{E}_x} + \mathscr{I}_1|_{\mathscr{E}_x}\Big) \colon \mathscr{E}_x \to \mathscr{E}_{x,1}, \quad \pi_{x,-1} = \frac{1}{2}\Big(\mathrm{id}|_{\mathscr{E}_x} - \mathscr{I}_1|_{\mathscr{E}_x}\Big) \colon \mathscr{E}_x \to \mathscr{E}_{x,-1}.$$

We first define the involutions on the subspace $\mathscr{E}_{x,-1}$. For $\xi \in \mathscr{E}_{x,-1}$ the second involution $\mathscr{I}_2$ is defined by the formula

$$\mathscr{I}_2\xi(t) = (-1)^{\lfloor t+1/2 \rfloor} J(\mathscr{I}_2 x(t))\xi(t + 1/2 - \lfloor t+1/2 \rfloor)) \in \mathscr{E}_{\mathscr{I}_2 x}.$$

To see that $\mathscr{I}_2\xi$ satisfies condition (i), i.e. maps the tangent space of the path space to itself, we have to check that if ξ is in $W^{1,p}$ and satisfies the Lagrangian boundary conditions, then $\mathscr{I}_2\xi$ satisfies these conditions, too. Since ξ lies in the eigenspace of the eigenvalue -1 of the first involution $\mathscr{I}_1$, it follows that

$$\xi(1/2) = -dR(x(1/2))\xi(1/2)$$

and using anticommutativity of the almost complex structure J with the antisymplectic involution R together with the fact that the Lagrangian L equals the fixpoint set of R it follows that

$$J(x(1/2))\xi(1/2) \in T_{x(1/2)}L$$

and hence $\mathscr{I}_2\xi$ satisfies the required boundary condition. To prove that $\mathscr{I}_2\xi \in W^{1,p}$ one has to check continuity at the point $t = 1/2$. This is done similarly as above by noting that

$$\xi(0) = -\xi(1) \in T_{x(0)}L = T_{x(1)}L.$$

Moreover, a straightforward calculation shows that

$$\mathscr{I}_1(\mathscr{I}_2\xi) = -\mathscr{I}_2\xi$$

and hence

$$\mathscr{I}_2\xi \in \mathscr{E}_{\mathscr{I}_2 x,-1}.$$

Now assume that $x \in \mathscr{P}_m$ for some integer $m > 2$. To define the involutions $\mathscr{I}_3, \ldots, \mathscr{I}_m$ on $\mathscr{E}_{x,-1}$ we first consider the following linear maps

$$\mathscr{L}_k \colon \mathscr{E}_{x,-1} \to \mathscr{E}_{\mathscr{I}_{k+2}x,-1}, \quad k \in \{1, \ldots, m-2\}$$

defined by

$$\mathscr{L}_k\xi(t) = \sum_{j=0}^{2^k-1} (-1)^{\lfloor \frac{j}{2^{k-1}} \rfloor + \lfloor t + \frac{1}{2^{k+1}} + \frac{j}{2^k} \rfloor} \xi\left(t + \frac{1}{2^{k+1}} + \frac{j}{2^k} - \lfloor t + \frac{1}{2^{k+1}} + \frac{j}{2^k} \rfloor\right).$$

Lemma 3.2 *The maps $\mathscr{L}_k$ for $1 \leq k \leq m-2$ are well-defined, i.e. $\mathscr{I}_1(\mathscr{L}_k\xi) = -\mathscr{L}_k\xi$. They have the following properties.*

(i) *They leave the tangent bundle of the path space invariant, i.e. for $T_{x,-1}\mathscr{P} := T_x\mathscr{P} \cap \mathscr{E}_{x,-1}$ we have*

$$\mathscr{L}_k(T_{x,-1}\mathscr{P}) = T_{\mathscr{I}_{k+2}x,-1}\mathscr{P}.$$

(ii) *They commute with each other and with $\mathscr{I}_2|_{\mathscr{E}_{x,-1}}$.*

(iii) *Setting $\mathscr{L}_0 :=$ id their squares can be determined recursively from the formula*

$$\mathscr{L}_{k+1}^2 = 2\left(\mathscr{L}_k^2 + \mathscr{L}_k\left(\sum_{i=0}^{k-1} \mathscr{L}_i\right)\right). \tag{3}$$

(iv) *If $p = 2$ then $\mathscr{L}_k$ is selfadjoint with respect to the L^2-inner product*

$$\langle \xi, \eta \rangle = \int_0^1 \xi(t)\eta(t)dt.$$

(v) *The maps $\mathscr{L}_k$ are injective.*

Proof: To prove that the maps $\mathscr{L}_k$ are well defined we calculate

$$
\begin{aligned}
& R^*\mathscr{L}_k\xi(1-t) \\
= & \sum_{j=0}^{2^k-1}(-1)^{\lfloor \frac{j}{2^{k-1}}\rfloor+\lfloor 1-t+\frac{1}{2^{k+1}}+\frac{j}{2^k}\rfloor} \\
& \cdot R^*\xi\Big(1-t+\frac{1}{2^{k+1}}+\frac{j}{2^k}-\lfloor 1-t+\frac{1}{2^{k+1}}+\frac{j}{2^k}\rfloor\Big) \\
= & -\sum_{j=0}^{2^k-1}(-1)^{\lfloor \frac{j}{2^{k-1}}\rfloor+\lfloor 1-t+\frac{1}{2^{k+1}}+\frac{j}{2^k}\rfloor} \\
& \cdot\xi\Big(t-\frac{1}{2^{k+1}}-\frac{j}{2^k}+\lfloor 1-t+\frac{1}{2^{k+1}}+\frac{j}{2^k}\rfloor\Big) \\
= & -\sum_{i=0}^{2^k-1}(-1)^{\lfloor 2-\frac{i}{2^{k-1}}-\frac{1}{2^{k-1}}\rfloor+\lfloor 1-t+\frac{1}{2^{k+1}}+1-\frac{i}{2^k}-\frac{1}{2^k}\rfloor} \\
& \cdot\xi\Big(t-\frac{1}{2^{k+1}}-1+\frac{i}{2^k}+\frac{1}{2^k}+\lfloor 1-t+\frac{1}{2^{k+1}}+1-\frac{i}{2^k}-\frac{1}{2^k}\rfloor\Big) \\
= & -\sum_{i=0}^{2^k-1}(-1)^{\lfloor -\frac{i}{2^{k-1}}-\frac{1}{2^{k-1}}\rfloor+\lfloor -t-\frac{1}{2^{k+1}}-\frac{i}{2^k}\rfloor} \\
& \cdot\xi\Big(t+\frac{1}{2^{k+1}}+\frac{i}{2^k}+1+\lfloor -t-\frac{1}{2^{k+1}}-\frac{i}{2^k}\rfloor\Big) \\
= & -\sum_{i=0}^{2^k-1}(-1)^{-\lfloor \frac{i}{2^{k-1}}-\frac{1}{2^{k-1}}\rfloor-1-\lfloor t+\frac{1}{2^{k+1}}+\frac{i}{2^k}\rfloor-1} \\
& \cdot\xi\Big(t+\frac{1}{2^{k+1}}+\frac{i}{2^k}-\lfloor t+\frac{1}{2^{k+1}}+\frac{i}{2^k}\rfloor\Big) \\
= & -\mathscr{L}_k\xi(t).
\end{aligned}
$$

The second last equality only holds in the case, when $t \neq \frac{j}{2^k}+\frac{1}{2^{k+1}}$ for $j \in \{0,\ldots,2^k-1\}$. However, the equality above still holds for ξ in the L^p-sense.

To prove assertion (i), one has to show that if $\xi \in T_{x,-1}\mathscr{P}$, then $\mathscr{L}_k\xi$ satisfies the Lagrangian boundary condition and is continuous at the points $t=\frac{1}{2^{k+1}}+\frac{j}{2^k}$ for $j \in \{0,\ldots 2^k-1\}$. To prove the boundary conditions one checks that

$$dR\mathscr{L}_k\xi(0)=\mathscr{L}_k\xi(0), \quad dR\mathscr{L}_k\xi(1)=\mathscr{L}_k\xi(1).$$

To prove continuity one uses

$$\xi(0)=-\xi(1)$$

which follows from the assumption that $\xi \in T_{x,-1}\mathscr{P}$.

To prove assertion (ii) one checks using the formula

$$\lfloor r-\lfloor s\rfloor\rfloor=\lfloor r\rfloor-\lfloor s\rfloor$$

that

$$
\begin{aligned}
&\mathscr{L}_k\mathscr{L}_\ell\xi(t)\\
= &\sum_{j=0}^{2^k-1}\sum_{i=0}^{2^\ell-1}(-1)^{\lfloor\frac{j}{2^{k-1}}\rfloor+\lfloor\frac{i}{2^{\ell-1}}\rfloor+\lfloor t+\frac{1}{2^{k+1}}+\frac{1}{2^{\ell+1}}+\frac{j}{2^k}+\frac{i}{2^\ell}\rfloor}\\
&\quad\cdot\xi\left(t+\frac{1}{2^{k+1}}+\frac{1}{2^{\ell+1}}+\frac{j}{2^k}+\frac{i}{2^\ell}-\lfloor t+\frac{1}{2^{k+1}}+\frac{1}{2^{\ell+1}}+\frac{j}{2^k}+\frac{i}{2^\ell}\rfloor\right).
\end{aligned}
$$

This formula is symmetric in k and ℓ and hence commutativity follows. In a similar way one proves commutativity with $\mathscr{I}_2|_{\mathscr{E}_{x,-1}}$.

It is straightforward to check that (3) holds if $k=0$. We now assume $k\geq 1$. Using the formula

$$
\sum_{i=0}^{k-1}\mathscr{L}_i\xi(t)=\sum_{\substack{\ell=0\\ \ell\neq 2^{k-1}}}^{2^k-1}(-1)^{\lfloor\frac{\ell}{2^{k-1}}\rfloor+\lfloor t+\frac{\ell}{2^k}\rfloor}\xi\left(t+\frac{\ell}{2^k}-\lfloor t+\frac{\ell}{2^k}\rfloor\right)
$$

we deduce that

$$
\begin{aligned}
\mathscr{L}_k\left(\sum_{i=0}^{k-1}\mathscr{L}_i\right)\xi(t) &= \sum_{\substack{\ell=0\\ \ell\neq 2^{k-1}}}^{2^k-1}\sum_{j=0}^{2^k-1}(-1)^{\lfloor\frac{\ell}{2^{k-1}}\rfloor+\lfloor\frac{j}{2^{k-1}}\rfloor+\lfloor t+\frac{\ell}{2^k}+\frac{j}{2^k}+\frac{1}{2^{k+1}}\rfloor}\\
&\quad\cdot\xi\left(t+\frac{\ell}{2^k}+\frac{j}{2^k}+\frac{1}{2^{k+1}}-\lfloor t+\frac{\ell}{2^k}+\frac{j}{2^k}+\frac{1}{2^{k+1}}\rfloor\right).
\end{aligned}
$$

We calculate

$$
\begin{aligned}
&\left(\mathscr{L}_{k+1}^2-2\mathscr{L}_k^2-2\mathscr{L}_k\left(\sum_{i=0}^{k-1}\mathscr{L}_i\right)\right)\xi(t)\\
= &\sum_{j=0}^{2^{k+1}-1}\sum_{i=0}^{2^{k+1}-1}(-1)^{\lfloor\frac{j}{2^k}\rfloor+\lfloor\frac{i}{2^k}\rfloor+\lfloor t+\frac{1+j+i}{2^{k+1}}\rfloor}\\
&\quad\cdot\xi\left(t+\frac{1+j+i}{2^{k+1}}-\lfloor t+\frac{1+j+i}{2^{k+1}}\rfloor\right)\\
&-2\sum_{j=0}^{2^k-1}\sum_{i=0}^{2^k-1}(-1)^{\lfloor\frac{j}{2^{k-1}}\rfloor+\lfloor\frac{i}{2^{k-1}}\rfloor+\lfloor t+\frac{1+j+i}{2^k}\rfloor}\\
&\quad\cdot\xi\left(t+\frac{1+j+i}{2^k}-\lfloor t+\frac{1+j+i}{2^k}\rfloor\right)\\
&-2\sum_{\substack{j=0\\ j\neq 2^{k-1}}}^{2^k-1}\sum_{i=0}^{2^k-1}(-1)^{\lfloor\frac{j}{2^{k-1}}\rfloor+\lfloor\frac{i}{2^{k-1}}\rfloor+\lfloor t+\frac{1+2j+2i}{2^{k+1}}\rfloor}\\
&\quad\cdot\xi\left(t+\frac{1+2j+2i}{2^{k+1}}-\lfloor t+\frac{1+2j+2i}{2^{k+1}}\rfloor\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \sum_{j=0}^{2^{k+1}-1} \sum_{i=0}^{2^{k+1}-1} (-1)^{\lfloor \frac{j}{2^k} \rfloor + \lfloor \frac{i}{2^k} \rfloor + \lfloor t + \frac{1+j+i}{2^{k+1}} \rfloor} \\
& \quad \cdot \xi\left(t + \frac{1+j+i}{2^{k+1}} - \lfloor t + \frac{1+j+i}{2^{k+1}} \rfloor \right) \\
& - \sum_{j=0}^{2^k-1} \sum_{i=0}^{2^k-1} (-1)^{\lfloor \frac{2j+1}{2^k} \rfloor + \lfloor \frac{2i}{2^k} \rfloor + \lfloor t + \frac{1+2j+1+2i}{2^{k+1}} \rfloor} \\
& \quad \cdot \xi\left(t + \frac{1+2j+1+2i}{2^{k+1}} - \lfloor t + \frac{1+2j+1+2i}{2^{k+1}} \rfloor \right) \\
& - \sum_{j=0}^{2^k-1} \sum_{i=0}^{2^k-1} (-1)^{\lfloor \frac{2j}{2^k} \rfloor + \lfloor \frac{2i+1}{2^k} \rfloor + \lfloor t + \frac{1+2j+2i+1}{2^{k+1}} \rfloor} \\
& \quad \cdot \xi\left(t + \frac{1+2j+2i+1}{2^{k+1}} - \lfloor t + \frac{1+2j+2i+1}{2^{k+1}} \rfloor \right) \\
& - \sum_{j=0}^{2^k-1} \sum_{i=0}^{2^k-1} (-1)^{\lfloor \frac{2j}{2^k} \rfloor + \lfloor \frac{2i}{2^k} \rfloor + \lfloor t + \frac{1+2j+2i}{2^{k+1}} \rfloor} \\
& \quad \cdot \xi\left(t + \frac{1+2j+2i}{2^{k+1}} - \lfloor t + \frac{1+2j+2i}{2^{k+1}} \rfloor \right) \\
& - \sum_{j=1}^{2^k} \sum_{i=0}^{2^k-1} (-1)^{\lfloor \frac{2j-1}{2^k} \rfloor + \lfloor \frac{2i+1}{2^k} \rfloor + \lfloor t + \frac{1+2j-1+2i+1}{2^{k+1}} \rfloor} \\
& \quad \cdot \xi\left(t + \frac{1+2j-1+2i+1}{2^{k+1}} - \lfloor t + \frac{1+2j-1+2i+1}{2^{k+1}} \rfloor \right) \\
= \quad & 0
\end{aligned}
$$

This proves (3).

To prove selfadjointness we calculate

$$
\begin{aligned}
& \int_0^1 \mathscr{L}_k\xi(t)\eta(t)dt \\
= & \sum_{j=0}^{2^k-1}\int_0^1 (-1)^{\lfloor \frac{j}{2^{k-1}}\rfloor + \lfloor t+\frac{1}{2^{k+1}}+\frac{j}{2^k}\rfloor} \\
& \cdot\xi\left(t+\frac{1}{2^{k+1}}+\frac{j}{2^k}-\lfloor t+\frac{1}{2^{k+1}}+\frac{j}{2^k}\rfloor\right)\eta(t)dt \\
= & \sum_{j=0}^{2^k-1}\int_0^1 (-1)^{\lfloor \frac{j}{2^{k-1}}\rfloor + \lfloor s-\frac{1}{2^{k+1}}-\frac{j}{2^k}-\lfloor s-\frac{1}{2^{k+1}}-\frac{j}{2^k}\rfloor+\frac{1}{2^{k+1}}+\frac{j}{2^k}\rfloor} \\
& \cdot\xi(s)\eta\left(s-\frac{1}{2^{k+1}}-\frac{j}{2^k}-\rfloor s-\frac{1}{2^{k+1}}-\frac{j}{2^k}\rfloor\right)ds \\
= & \sum_{j=0}^{2^k-1}\int_0^1 (-1)^{\lfloor \frac{j}{2^{k-1}}\rfloor + \lfloor s-\lfloor s-\frac{1}{2^{k+1}}-\frac{j}{2^k}\rfloor\rfloor} \\
& \cdot\xi(s)\eta\left(s-\frac{1}{2^{k+1}}-\frac{j}{2^k}-\rfloor s-\frac{1}{2^{k+1}}-\frac{j}{2^k}\rfloor\right)ds \\
= & \sum_{i=0}^{2^k-1}\int_0^1 (-1)^{\lfloor 2-\frac{1}{2^{k-1}}-\frac{i}{2^{k-1}}\rfloor + \lfloor s-\lfloor s-\frac{1}{2^{k+1}}-1+\frac{1}{2^k}+\frac{i}{2^k}\rfloor\rfloor} \\
& \cdot\xi(s)\eta\left(s-\frac{1}{2^{k+1}}-1+\frac{1}{2^k}+\frac{i}{2^k}-\lfloor s-\frac{1}{2^{k+1}}-1+\frac{1}{2^k}+\frac{i}{2^k}\rfloor\right) \\
= & \sum_{i=0}^{2^k-1}\int_0^1 (-1)^{\lfloor -\frac{1}{2^{k-1}}-\frac{i}{2^{k-1}}\rfloor + \lfloor s\rfloor-\lfloor s+\frac{1}{2^{k+1}}+\frac{i}{2^k}\rfloor+1} \\
& \cdot\xi(s)\eta\left(s+\frac{1}{2^{k+1}}+\frac{i}{2^k}-\lfloor s+\frac{1}{2^{k+1}}+\frac{i}{2^k}\rfloor\right) \\
= & \sum_{i=0}^{2^k-1}\int_0^1 (-1)^{-\lfloor \frac{1}{2^{k-1}}+\frac{i}{2^{k-1}}\rfloor - \lfloor s+\frac{1}{2^{k+1}}+\frac{i}{2^k}\rfloor} \\
& \cdot\xi(s)\eta\left(s+\frac{1}{2^{k+1}}+\frac{i}{2^k}-\lfloor s+\frac{1}{2^{k+1}}+\frac{i}{2^k}\rfloor\right) \\
= & \int_0^1 \xi(s)\mathscr{L}_k\eta(s)ds
\end{aligned}
$$

To prove injectivity we observe that using the formula

$$
\begin{aligned}
\mathscr{L}_k\xi(t+\frac{i}{2^k}) \quad = \quad & \sum_{\ell=0}^{2^k-1}(-1)^{\lfloor\frac{\ell-i}{2^{k-1}}\rfloor+\lfloor\frac{\ell-i}{2^k}\rfloor+\lfloor t+\frac{1}{2^{k+1}}+\frac{\ell}{2^k}\rfloor} \\
& \cdot\xi\left(t+\frac{1}{2^{k+1}}+\frac{\ell}{2^k}-\lfloor t+\frac{1}{2^{k+1}}+\frac{\ell}{2^k}\rfloor\right)
\end{aligned}
$$

for $i \in \{0, \ldots, 2^k - 1\}$ and $0 \leq t < 1/2^k$ it suffices to show that the matrix $A(t) \in \mathbb{R}^{2^k \times 2^k}$ whose entries are given by

$$A_{i,\ell}(t) = (-1)^{\lfloor \frac{\ell - i}{2^{k-1}} \rfloor + \lfloor \frac{\ell - i}{2^k} \rfloor + \lfloor t + \frac{1}{2^{k+1}} + \frac{\ell}{2^k} \rfloor}$$

is nondegenerate. Using the elementary fact that the determinant of a matrix is invariant if one subtracts from a row a different row together with the formulas

$$|A_{i+1,\ell}(t) - A_{i,\ell}(t)| = 2\delta_{i+2^{k-1},\ell}, \quad i \in \{0, \ldots, 2^{k-1} - 1\}$$

$$|A_{i+1,\ell}(t) - A_{i,\ell}(t)| = 2\delta_{i-2^{k-1},\ell}, \quad i \in \{2^{k-1} - 1, \ldots, 2^k - 2\}$$

$$|A_{0,\ell}(t) - A_{2^k-1,\ell}(t)| = 2\delta_{2^{k-1}-1,\ell}$$

one concludes that

$$|\det(A)| = 2^{(2^k)}.$$

This proves injectivity and hence the lemma follows. $\square$

It follows from (3) and from (ii) in the preceding lemma that for each $k \in \{1, \ldots, m-2\}$ the spectrum of $\mathscr{L}_k^2 \colon \mathscr{E}_{x,-1} \to \mathscr{E}_{x,-1}$ consists of a finite number of eigenvalues. These eigenvalues do not depend on x and can be computed recursively from (3). For $\mathscr{L}_1^2$ the unique eigenvalue is given by

$$\lambda_1^1 = 2$$

for $\mathscr{L}_2^2$ the two eigenvalues are given by

$$\lambda_1^2 = 4 + 2 \cdot \sqrt{2}, \quad \lambda_2^2 = 4 - 2 \cdot \sqrt{2}$$

and for $\mathscr{L}_3^2$ the four eigenvalues are given by

$$\begin{aligned}
\lambda_1^3 &= 2\left(4 + 2 \cdot \sqrt{2} + \sqrt{4 + 2 \cdot \sqrt{2}}\left(1 + \sqrt{2}\right)\right) \\
\lambda_2^3 &= 2\left(4 - 2 \cdot \sqrt{2} + \sqrt{4 - 2 \cdot \sqrt{2}}\left(1 - \sqrt{2}\right)\right) \\
\lambda_3^3 &= 2\left(4 + 2 \cdot \sqrt{2} - \sqrt{4 + 2 \cdot \sqrt{2}}\left(1 + \sqrt{2}\right)\right) \\
\lambda_4^3 &= 2\left(4 - 2 \cdot \sqrt{2} - \sqrt{4 - 2 \cdot \sqrt{2}}\left(1 - \sqrt{2}\right)\right).
\end{aligned}$$

We claim that all the eigenvalues of all the maps $\mathscr{L}_k$ for $1 \leq k \leq m-2$ are positive. It follows from assertion (v) in the preceding lemma that no eigenvalue of $\mathscr{L}_k$ is zero. To see that the eigenvalues are nonnegative we use the fact that they are independent of the path x and the index of integrability p. For $k < m$ the map $\mathscr{L}_k$ maps $\mathscr{E}_{x,-1}$ to itself and moreover $\mathscr{L}_k$ is selfadjoint with respect

to the L^2-inner product $L^2([0,1],T^*M)$ by assertion (iv). It follows that all the eigenvalues of $\mathscr{L}_k$ are real and hence all eigenvalues of $\mathscr{L}_k^2$ are nonnegative. Using independence of the eigenvalues from x and p the claim follows.

For $1 \leq k \leq m-2$ denote by $\Pi_{k,\lambda}$ for $\lambda \in \sigma(\mathscr{L}_k^2)$ the projection to the eigenspace of λ in $\mathscr{E}_{x,-1}$. For $3 \leq k \leq m$ we now extend the involution $\mathscr{I}_k$ to $\mathscr{E}_{x,-1}$ by the formula

$$\mathscr{I}_k\xi := \sum_{\lambda\in\sigma(\mathscr{L}_{k-2}^2)} \frac{1}{\sqrt{\lambda}}\mathscr{L}_{k-2}\circ\Pi_{k-2,\lambda}\xi, \quad \xi \in \mathscr{E}_{x,-1}.$$

It remains to extend the involutions also the $\mathcal{E}_{x,1}$ the eigenspace to the eigenvalue 1 of the first involution. To do that we introduce the maps

$$\mathscr{H}_k\colon \mathscr{P}_{k+1} \to \mathscr{P}_k, \quad \mathscr{D}_k\colon \mathscr{P}_k \to \mathscr{P}_{k+1}, \quad k \in \mathbb{N}$$

by

$$\mathscr{H}_k x(t) := x(t/2), \quad 0 \leq t \leq 1,\ x \in \mathscr{P}_{k+1}$$

and

$$\mathscr{D}_k x(t) = \begin{cases} x(2t) & 0 \leq t \leq 1/2 \\ R(x(2-2t)) & 1/2 \leq t \leq 1. \end{cases}$$

We extend these maps in the obvious way to bundle maps

$$\mathscr{H}_k\colon \mathscr{E}_x \to \mathscr{E}_{\mathscr{H}_k(x)}, \quad x \in \mathscr{P}_{k+1}$$

and

$$\mathscr{D}_k\colon \mathscr{E}_x \to \mathscr{E}_{\mathscr{D}_k(x)}, \quad x \in \mathscr{P}_k$$

by setting

$$\mathscr{H}_k\xi(t) := \xi(t/2), \quad 0 \leq t \leq 1,\ \xi \in \mathscr{E}_x$$

and

$$\mathscr{D}_k\xi(t) = \begin{cases} \xi(2t) & 0 \leq t \leq 1/2 \\ R^*\xi(2-2t) & 1/2 \leq t \leq 1. \end{cases}$$

Note that

$$\mathscr{H}_k \circ \mathscr{D}_k = \mathrm{id}|_{\mathscr{E}_x}.$$

We now define recursively for $\xi \in \mathscr{E}_{x,1}$

$$\mathscr{I}_{k+1}\xi := \mathscr{D}_k \circ \mathscr{I}_k \circ \mathscr{H}_k\xi, \quad k \in \{1,\dots,m-1\}.$$

Remark 3.3 *Recall that the vertical differential of the section $\mathcal{F}$ is given by*

$$D_u\xi = \partial_s\xi + J(u)\partial_t\xi + \nabla_\xi J(u)\partial_t u$$

where $u \in \mathcal{F}^{-1}(0)$, $\xi \in T_u\mathcal{B}$, and ∇ denotes the Levi-Civita connection of the metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Observe that the last term does anticommute with the almost complex structure J. The compact operator in assertion (ii) of Theorem 3.1 is given by

$$Q_u\xi = \nabla_\xi J(u)\partial_t u, \quad \xi \in T_u\mathcal{B}.$$

If the almost complex structure is integrable, i.e. $\nabla J = 0$, then Q_u vanishes and D_u commutes with J and hence interchanges the two involutions.

3.3 Kuranishi structures

We recall here the definition of Kuranishi structure as defined in [FOn], see also [FOOO]. Our definition will be less general than the one in [FOn] since our Kuranishi neighbourhoods consist of manifolds instead of orbifolds, which is sufficient for our purposes. Let X be a compact topological Hausdorff space. A Kuranishi structure assigns (V_p, E_p, ψ_p, s_p) to each $p \in X$ and $(V_{pq}, \hat{\phi}_{pq}, \phi_{pq})$ to points $p, q \in X$ which are close to each other. They are required to satisfy the following properties:

K1: V_p is a smooth manifold, and E_p is a smooth vector bundle on it.

K2: s_p is a continuous section of E_p.

K3: ψ_p is a homeomorphism from $s_p^{-1}(0)$ to a neighbourhood of $p \in X$.

K4: $V_{pq}, \hat{\phi}_{pq}, \phi_{pq}$ are defined if $q \in \psi_p(s_p^{-1}(0))$.

K5: V_{pq} is an open subset of V_q containing $\psi_q^{-1}(q)$.

K6: $(\hat{\phi}_{pq}, \phi_{pq})$ is a map of vector bundles $E_q|_{V_{pq}} \to E_p$.

K7: $\hat{\phi}_{pq} s_q = s_p \phi_{pq}$.

K8: $\psi_q = \psi_p \phi_{pq}$.

K9: If $r \in \psi_q(s_q^{-1}(0) \cap V_{pq})$, then

$$\hat{\phi}_{pq} \circ \hat{\phi}_{qr} = \hat{\phi}_{pr}$$

in a neighbourhood of $\psi_r^{-1}(r)$.

K10: $\dim V_p - \operatorname{rank} E_p$ does only depend on the connected component of X in which p lies and is called the *virtual dimension* of the Kuranishi structure of the connected component of p.

Following [FOn] we will say that our Kuranishi structure has a *tangent bundle* if there exists a family of isomorphisms

$$\Phi_{pq} \colon N_{V_p} V_q \cong E_p / E_q$$

satisfying the usual compatibility conditions. Here $N_{V_p} V_q$ denotes the normal bundle of V_q in V_p.

Definition 3.4 *We say that a compact topological Hausdorff space X and a sequence of continuous involutions $\{I_k\}_{1 \le k \le m}$ defined on closed subspaces $X_m \subset X_{m-1} \cdots \subset X_1 = X$ of X are of* **Arnold-Givental type***, if the following holds.*

(i) *The domain of the first involution X_1 is the whole space X, and the domain X_k of I_k for $2 \le k \le m$ is the fixpoint set of the previous involution, i.e.*

$$X_k = \operatorname{Fix}(I_{k-1}).$$

(ii) *The last involution acts freely, i.e.*

$$\mathrm{Fix}(I_m) = \emptyset.$$

Definition 3.5 *We say that a space of Arnold-Givental type $(X, \{I_k\}_{1 \le k \le m})$ has a* **Kuranishi structure** *if X admits a Kuranishi structure in the sense of Fukaya-Ono, such that in addition for each $p \in X_k$ there exist involutions $I_{p,j}$ for $1 \le j \le k$ where $I_{p,1}$ is defined on $V_{p,1} := V_p$ and $I_{p,j}$ is defined on $V_{p,j} := \mathrm{Fix}(I_{p,j-1})$ for $2 \le j \le k$, and extensions of $I_{p,j}$ to smooth bundle involutions*

$$\hat{I}_{p,j} \colon E_p|_{V_{p,j}} \to E_p|_{V_{p,j}},$$

where the following conditions are satisfied.

(i) *If $p \in X_k \setminus X_{k+1}$ and $q \in \psi_p(s_p^{-1}(0)) \cap X_j$, then $j \le k$, $I_{p,k}$ acts freely on $V_{p,k}$, and*

$$I_j(q) = \psi_p \circ I_{p,j} \circ \psi_p^{-1}(q).$$

(ii) *If $p, q \in X$ are close enough, $x \in V_{pq} \cap V_{q,j}$, and $\xi \in (E_q)_x$ then*

$$I_{p,j}(\phi_{pq}(x)) = \phi_{pq}(I_{q,j}(x))$$

and

$$\hat{I}_{p,j} \circ \hat{\phi}_{pq}\xi = \hat{\phi}_{pq} \circ \hat{I}_{q,j}\xi.$$

(iii) *The bundle involutions commute on their common domain of definition, i.e. if $x \in V_{p,j}$ and $\xi \in (E_p)_x$, then*

$$\hat{I}_{p,\ell} \circ \hat{I}_{p,j}\xi = \hat{I}_{p,j} \circ \hat{I}_{p,\ell}\xi$$

for $1 \le \ell \le j$.

Remark 3.6 *Assume that $(X, \{I_k\}_{1 \le k \le m})$ has Arnold-Givental type. Then also $(X_j, \{I_k\}_{j \le k \le m})$ for $1 \le j \le m$ has Arnold-Givental type. Moreover, if X has a Kuranishi structure, then also X_j has a Kuranishi structure. A Kuranishi neighbourhood is constructed in the following way. For $p \in X_k$ with $k \ge j$ take*

$$V_{p,j} = \mathrm{Fix}(I_{p,j-1}|_{V_p})$$

and as obstruction bundle take the intersection of the eigenspaces to the eigenvalue 1 of the previous involutions, i.e.

$$E_{p,j} := \bigcap_{1 \le i \le j-1} \ker(\hat{I}_{p,i}|_{E_p|V_p^j} - \mathrm{id}|_{E_p|V_p^j}).$$

The other ingredients of the Kuranishi structure are then given by the obvious restrictions. Note that since the involutions commute on their common domain of definition $E_{p,j}$ is invariant under $\hat{I}_{p,i}$ for $j \le i \le k$.

Definition 3.7 *We say that a space* $(X, \{I_k\}_{1 \le k \le m})$ *of Arnold-Givental which admits a Kuranishi structure has a* **tangent bundle** *if the Kuranishi spaces* X_j *admit a tangent bundle in the sense of Fukaya-Ono and the isomorphisms* $\Phi_{pq,j}$ *for* $p, q \in X_j$ *close enough are obtained by restriction of* $\Phi_{pq,1} = \Phi_{pq}$ *for* $1 \le j \le m$.

In order to do useful perturbation theory we have to extend the involutions to the tangent bundle. Since by assertion (ii) of Theorem 3.1 the vertical differential of the section $\mathcal{F}$ commutes with the involutions only modulo a compact operator the tangent bundle will in general only admit a "stable" Arnold-Givental structure. A similar phenomenon appeared in [FOn] where the Kuranishi structure in general only admitted a stable almost complex structure.

We first have to recall the following terminology from [FOn]. A tuple $((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))$ is a bundle system over the Kuranishi space $X = (X, (V_p, E_p, \psi_p, s_p))$ if $F_{1,p}$ and $F_{2,p}$ are two vector bundles over V_p for every $p \in X$, $\Phi_{1,pq} \colon F_{1,q} \to F_{1,p}|_{V_q}$ and $\Phi_{2,pq} \colon F_{2,q} \to F_{2,p}|_{V_q}$ are embeddings for q sufficiently close to p, and

$$\Phi_{pq} \colon \frac{F_{1,p}|_{V_q}}{F_{1,q}} \to \frac{F_{2,p}|_{V_q}}{F_{2,q}}$$

are isomorphisms of vector bundles. These maps are required to satisfy some compatibility conditions. Moreover, there is some obvious notion of isomorphism, Whitney sum, tensor product etc. for bundle systems. We refer the reader to [FOn] for details.

If a Kuranishi structure has a tangent bundle, i.e. a family of isomorphisms $\Phi_{pq} \colon N_{V_p} V_q \cong E_p / E_q$, then one can define a bundle system

$$TX = (TV_p, E_p, \hat{\phi}_{pq}, d\phi_{pq}, \Phi_{pq})$$

over X. If the space X is of Arnold-Givental type and admits a tangent bundle, then we define the normal bundle

$$\begin{aligned} NX \quad = \quad & \{TV_{p,j}|_{V_{p,j+1}} / TV_{p,j+1}, \\ & E_{p,j}|_{V_{p,j+1}} / E_{p,j+1}, \\ & d\phi_{pq,j}|_{V_{pq,j+1}} / d\phi_{pq,j+1}, \\ & \hat{\phi}_{pq,j}|_{V_{pq,j+1}} / \hat{\phi}_{pq,j+1}, \\ & \Phi_{pq,j}|_{V_{pq,j+1}} / \Phi_{pq,j+1}\}_{1 \le j \le m-1} \end{aligned}$$

as a bundle system over $\bigcup_{j=1}^{m-1} X_{j+1}$. The real K-group $KO(X)$ of a space with Kuranishi structure X was defined in [FOn] as the quotient of the free abelian group generated by the set of all isomorphism classes of bundle systems modulo the relations

$$[((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq})) \oplus ((F'_{1,p}, F'_{2,p}), (\Phi'_{1,pq}, \Phi'_{2,pq}, \Phi'_{pq}))] =$$

$$[((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))] + [((F'_{1,p}, F'_{2,p}), (\Phi'_{1,pq}, \Phi'_{2,pq}, \Phi'_{pq}))]$$

$$[((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq}))] = 0$$
$$\text{if } ((F_{1,p}, F_{2,p}), (\Phi_{1,pq}, \Phi_{2,pq}, \Phi_{pq})) \text{ is trivial.}$$

If the Kuranishi structure has a tangent bundle $[TX]$ denotes the class of TX in $KO(X)$. If the Kuranishi structure is of Arnold-Givental type and admits a tangent bundle $[NX]$ denotes the class of NX in $\bigoplus_{j=2}^m KO(X_j)$.

Definition 3.8 *Assume that $(X, \{I_k\}_{1\le k\le m})$ is a space of Arnold-Givental type which admits a Kuranishi structure $(X, (V_p, E_p, s_p, \psi_p, \phi_{pq}, \hat{\phi}_{pq}))$ and assume further that $((F_{1,p}, F_{2,p}), (\Phi_{1;pq}, \Phi_{2;pq}, \Phi_{pq}))$ is a bundle system on it. We say that $((F_{1,p}, F_{2,p}), (\Phi_{1;pq}, \Phi_{2;pq}, \Phi_{pq}))$ is a* **Bundle System of Arnold-Givental type** *if for every $p \in X_k$ there exist extensions of the involutions $I_{p,j}$ for $1 \le j \le k$ to smooth involutative bundle maps $\hat{I}_{1,p,j}\colon F_{1,p}|_{V_{p,j}} \to F_{1,p}|_{V_{p,j}}$ and $\hat{I}_{2,p,j}\colon F_{2,p}|_{V_{p,j}} \to F_{2,p}|_{V_{p,j}}$ such that the following conditions are satisfied.*

Compatibility: *The transition maps Φ_{pq}, $\Phi_{1;pq}$, and $\Phi_{2;pq}$ restricted to the domain of definition of the involutions interchange them.*

Commutativity: *The involutions commute on their common domain of definition.*

The Whitney sum of two bundle systems of Arnold-Givental type still has Arnold-Givental type and hence one can consider the K-group $K_{AG}(X)$ of bundle systems of Arnold-Givental type over X. There is an obvious map

$$K_{AG}(X) \to KO(X).$$

If X admits a tangent bundle then there is an obvious extension of $I_{p,j}$ to the bundle E_p given by $\hat{I}_{p,j}$. However there is no obvious extension of $I_{p,j}$ to TV_p in general. This motivates the following definition.

Definition 3.9 *Assume that $(X, \{I_k\}_{1\le k\le m})$ is a space of Arnold-Givental type which has a Kuranishi structure with tangent bundle. We say that X has a* **Kuranishi structure of Arnold-Givental type** *if the normal bundle NX is a bundle system of Arnold-Givental type in $\bigcup_{j=2}^m X_j$. We say that X has a* **Kuranishi structure of stable Arnold-Givental type** *if $[NX]$ is in the image of $\bigoplus_{j=2}^m K_{AG}(X_j) \to \bigoplus_{j=2}^m KO(X_j)$.*

Note that a Kuranishi structure of Arnold-Givental type is also a Kuranishi structure of stable Arnold-Givental type.

Theorem 3.10 *We assume that $(X, \{I_k\}_{1\le k\le m})$ has a Kuranishi structure of Arnold-Givental type whose virtual dimension is zero. Then for each $p \in X$, there exist smooth sections $\tilde{s}_p$ such that the following holds.*

(i) $\tilde{s}_p \circ \phi_{pq} = \hat{\phi}_{pq} \circ \tilde{s}_q$,

(ii) *Let $x \in V_{pq}$. Then the restriction of the differential of the composition of $\tilde{s}_p$ and the projection $E_p \to E_p/E_q$ coincides with the isomorphism $\Phi_{pq}\colon N_{V_p}V_q \cong E_p/E_q$.*

(iii) *The sections $\tilde{s}_p$ are transversal to* 0*, and if* $p \in X_k$ *then* $\tilde{s}_p^{-1}(0)$ *is invariant under* $I_{p,j}$ *for* $1 \le j \le k$*.*

Remark 3.11 *The sections $\tilde{s}_p$ will not necessarily be invariant under the involutions, only their zero set will be.*

We need the following lemma.

Lemma 3.12 *Assume that Y is a compact manifold, N and E are two vector bundle over Y and $U \subset N$ is an open neighbourhood of $Y \subset N$. Denote by $\pi_N \colon N \to Y$ the canonical projection. Then there exists a section $s \colon N \to \pi_N^* E$ which satisfies the following conditions.*

(i) *The zero section $s^{-1}(0)$ is invariant under the involutative bundle map on N defined by $(n, y) \mapsto (-n, y)$ for $y \in Y$ and $n \in N_y$.*

(ii) *Outside U the section s is invariant under the involutative bundle map of $\pi_N^* E$ given by $(e, n, y) \mapsto (-e, -n, y)$ for $y \in Y$, $n \in N_y$, and $e \in \pi_N^* E_{(n,y)}$.*

(iii) *The boundary of the set of transversal points has codimension at least one. More precisely, there exists a manifold Ω of dimension $\dim(\Omega) = \dim(Y) - \mathrm{rk}(E) + \mathrm{rk}(N) - 1$ and a smooth map $f \colon \Omega \to N$ such that the ω-limit set of the $\dim(Y) - \mathrm{rk}(E) + \mathrm{rk}(N)$-dimensional manifold of points of transversal intersection*

$$\mathscr{T} := \{n \in s^{-1}(0) : Ds(n) \text{ onto}\}$$

is contained in the image of Ω, i.e.

$$\bigcap_{K \subset \mathscr{T} \text{ compact}} \mathrm{cl}(\mathscr{T} \setminus K) \subset f(\Omega).$$

Proof: Choose a bundle map $\Phi \colon N \to E$ and define

$$Q := \{y \in Y : \dim(\ker\Phi(y)) > 0\}.$$

Let $\Psi \colon Y \to E$ be a section such that

$$\Psi|_Q = 0, \quad \Psi(y) \cap \Phi(N_y) = \{0\}, \quad \forall\, y \in Y.$$

Choose further a smooth cutoff function $\beta \colon N \to \mathbb{R}$ such that

$$\beta|_Y = 1, \quad \mathrm{supp}(\beta) \subset U.$$

Define a section $s \colon N \to \pi_N^* E$ by

$$s(n) := \beta(n) \pi_N^* \Psi(\pi_N(n)) + \pi_N^* \Phi(n), \quad n \in N.$$

Then s satisfies conditions (i) and (ii) and for generic choice of Ψ and Φ also condition (iii). This proves the lemma. $\square$

Proof of Theorem 3.10: We continue the notation of Remark 3.6. For $j \in \{1, \cdots, m\}$ denote by s_p^j for $p \in X_j$ the section from $V_{p,j}$ to $E_{p,j}$ which is induced from s_p. By induction on j from m to 1 we find sections $\tilde{s}_p^j$ of s_p^j which satisfy conditions (i) and (ii) of Theorem 3.10 as well invariance of the zero set under the involutions, but instead of the transversality condition we impose the condition that the boundary of the manifold of points of transversal intersection has codimension at least one.

(iii') For each $j \in \{1, \cdots, m\}$ and for each $p \in X_j$ there exists a manifold Ω_p^j of dimension $\dim(\Omega_p^j) = d_p^j - 1$ where d_p^j is the virtual dimension of the connected component of p of the Kuranishi structure of X_j and a smooth map $f_p^j \colon \Omega_p^j \to V_p^j$ such that the ω-limit set of the set of points of transversal intersection

$$\mathscr{T}_p^j := \{q \in (\tilde{s}_p^j)^{-1}(0) : D\tilde{s}_p^j(q) \ onto\}$$

is contained in the image of Ω_p^j, i.e.

$$\bigcap_{K \subset \mathscr{T}_p^j \ compact} \mathrm{cl}(\mathscr{T}_p^j \setminus K) \subset f_p^j(\Omega_p^j).$$

To prove the induction step we observe that by the assumption that the Kuranishi structure is of Arnold-Givental type it follows that the two vector bundles $TV_{p,j}|_{(s_p^{j+1})^{-1}(0)}/TV_{p,j+1}|_{(s_p^{j+1})^{-1}(0)}$ and $E_{p,j}|_{(s_p^{j+1})^{-1}(0)}/E_{p,j+1}|_{(s_p^{j+1})^{-1}(0)}$ induce bundles on the quotient $(s_p^{j+1})^{-1}(0)/I_{p,j+1}$. The induction step can now be concluded from Lemma 3.12. Since the Kuranishi structure of $X = X_1$ has virtual dimension zero it follows that for every $p \in X_1$ the set Ω_p^1 is empty and hence condition (iii) holds. This proves the theorem. □

We recall from [FOn] that if Y is a topological space and X is a space with Kuranishi structure, a *strongly continuous (smooth) map* $f : X \to Y$ is a family of continuous (smooth) maps $f_p \colon V_p \to Y$ for each $p \in X$ such that $f_p \circ \phi_{pq} = f_q$. If n is the virtual dimension of the Kuranishi structure we define the homology class

$$f([X]) \in H_n(Y; \mathbb{Z}_2)$$

as in [FOn]. We are now able to draw the following Corollary from Theorem 3.10.

Corollary 3.13 *Assume that $(X, \{I_k\}_{1 \leq k \leq m})$ has a Kuranishi neighbourhood of stable Arnold-Givental type whose virtual dimension is zero. Suppose further that Y is a topological space and $f \colon X \to Y$ is a strongly continuous map. Then*

$$f([X]) = 0 \in H_0(Y; \mathbb{Z}_2).$$

If the space $\mathcal{M}$ consisting of J-holomorphic disks whose boundary is mapped to the Lagrangian L is compact with respect to the Gromov topology, then $\mathcal{M}$

together with the involutions described in section 2 is of Arnold-Givental type. We next prove that under the compactness assumption $\mathcal{M}$ has a Kuranishi neighbourhood of Arnold-Givental type. In our proof we mainly follow [FOn]. The new ingredient is to choose the obstruction bundle in such a way that it is invariant under the involutions.

In general one cannot expect that $\mathcal{M}$ is compact due to the bubbling phenomenon. We hope that the approach pursued in this article together with the techniques developped in [FOn] and [FOOO] will allow us to show that the compactification of $\mathcal{M}$ by bubble trees has a Kuranishi neighbourhood of Arnold-Givental type.

Theorem 3.14 *Assume that $\mathcal{M}$ is compact with respect to the Gromov topology. Then $(\mathcal{M}, \{I_k\}_{1\leq k\leq m})$ admits a Kuranishi structure of stable Arnold-Givental type.*

Proof: We first choose local trivialisations of the vector bundle $\mathcal{E}$. More precisely, for each $q \in \mathcal{B}$ we choose an open neighbourhood $U_q \subset \mathcal{B}$ of q and a smooth family of Banach space isomorphisms

$$\mathcal{P}_q^p \colon \mathcal{E}_q \to \mathcal{E}_p, \quad p \in U_q$$

such that the following conditions are satisfied.

(T1) If $q \in \mathcal{B}_k \setminus \mathcal{B}_{k+1}$ for $k \in \mathbb{N}$, then U_q is invariant under I_j for $1 \leq j \leq k$ and I_k acts freely on U_q.

(T2) For $1 \leq j \leq k$ the trivialisations commute with the involutions, i.e.

$$\mathcal{P}_q^p \circ I_j^{\mathcal{E}} = I_j^{\mathcal{E}} \circ \mathcal{P}_{I_j q}^{I_j p}, \quad p \in U_q.$$

Now choose for every $p \in \mathcal{M}$ an open neighbourhood $\hat{V}_p \subset U_p$ of p in $\mathcal{B}$, choose a finite set $Q \subset \mathcal{M}$, for each $q \in Q$ a closed neighbourhood $\hat{U}_q \subset U_q$ of q in $\mathcal{B}$, and a finite dimensional subspace $\hat{E}_q \subset \mathcal{E}_q$ consisting of smooth sections with the following properties:

(i) For every $k \in \mathbb{N}$ the sets $\hat{V}_p$ for every $p \in \mathcal{M} \cap (\mathcal{B}_k \setminus \mathcal{B}_{k+1})$ and the set $\bigcup_{q\in Q\cap(\mathcal{B}_k\setminus\mathcal{B}_{k+1})} \hat{U}_q$ are invariant under I_j for $1 \leq j \leq k$ and I_k acts freely on them.

(ii) For $1 \leq k \leq m$ the family of vectorspaces $\bigcup_{q\in\mathcal{M}_k} \hat{E}_q$ is invariant under $I_k^{\mathcal{E}}$.

(iii) For $p \in \mathcal{B}$ let $Q_p := \{q \in Q : p \in \hat{U}_q\}$. Assume that $p \in \mathcal{M}$ and $p' \in \hat{V}_p$. Then the sum $\bigoplus_{q\in Q_p} \mathcal{P}_q^{p'} \hat{E}_q$ is direct, i.e. for every $q_0 \in Q_p$ we have $\mathcal{P}_{q_0}^{p'} \hat{E}_{q_0} \cap \sum_{q\in Q_p\setminus\{q_0\}} \mathcal{P}_q^{p'} \hat{E}_q = \emptyset$.

(iv) For $p \in \mathcal{M}$ and $p' \in \hat{V}_p$ the operator

$$\Pi_p^{p'} \circ D_{p'} \colon T_{p'}\mathcal{B} \to \mathcal{E}_{p'} / \bigoplus_{q\in Q_p} \mathcal{P}_q^{p'} \hat{E}_q$$

is surjective, where

$$\Pi_p^{p'} : \mathcal{E}_{p'} \to \mathcal{E}_{p'} / \bigoplus_{q \in Q_p} \mathcal{P}_q^{p'} \hat{E}_q$$

denotes the canonical projection.

Our first aim is to define the manifold V_p of p which occurs in the definition of Kuranishi structure. In our construction this manifold will be a small neighbourhood of zero in the kernel of the map $\Pi_p^p \circ D_p$. The size of this neighbourhood will depend on the domain of definition of the smooth injective evaluation maps

$$\mathrm{ev}_p \colon V_p \to \hat{V}_p$$

which we have to define first.

By (iv) we can choose a smooth family of uniformly bounded right inverses

$$R_p^{p'} : \mathcal{E}_{p'} / \bigoplus_{q \in Q_p} \mathcal{P}_q^{p'} \hat{E}_q \to T_{p'} \mathcal{B}$$

of $\Pi_p^{p'} \circ D_{p'}$, i.e.

$$\Pi_p^{p'} \circ D_{p'} \circ R_p^{p'} = \mathrm{id}|_{\mathcal{E}_{p'} / \bigoplus_{q \in Q_p} \mathcal{P}_q^{p'} \hat{E}_p}.$$

We need in addition some compatibility of the right inverses with the involutions. To state it we observe that it follows from condition (T2) on the local trivialisations together with conditions (i) and (ii) that for $p \in \mathcal{M}_k$ and $p' \in \hat{V}_p \cap \mathcal{B}_j$ for $j \leq k$ the family of vector spaces $\bigoplus_{q \in Q_p} \mathcal{P}_q^{p'} \hat{E}_q \cup \bigoplus_{q \in Q_{I_i p}} \mathcal{P}_{I_i q}^{I_i p'} \hat{E}_{I_i q}$ is invariant under I_i for $1 \leq i \leq j$. Hence the involutions $I_j^{\mathcal{E}}$ induce involutions

$$I_j^{\mathcal{E},p} \colon \bigcup_{p' \in \hat{V}_p \cap \mathcal{B}_j} \mathcal{E}_{p'} / \bigoplus_{q \in Q_p} \mathcal{P}_q^{p'} \hat{E}_q \to \bigcup_{p' \in \hat{V}_p \cap \mathcal{B}_j} \mathcal{E}_{p'} / \bigoplus_{q \in Q_p} \mathcal{P}_q^{p'} \hat{E}_q.$$

Using the fact that by (ii) of Theorem 3.1 the operator $I_k^{\mathcal{E}} \circ D_u - D_u \circ I_k^{T\mathcal{B}}$ vanishes on $T_u \mathcal{B}_k$ together with (iii) of Theorem 3.1, we can impose the following compatibility condition of the right inverse $R_p^{p'}$ and the involutions

$$R_p^{p'} \circ I_k^{\mathcal{E},p}|_{\bigcap_{j=1}^{k-1} \ker(I_j^{\mathcal{E},p} - \mathrm{id})} = I_k^{T\mathcal{B}} \circ R_p^{p'}|_{\bigcap_{j=1}^{k-1} \ker(I_j^{\mathcal{E},p} - \mathrm{id})}, \quad \forall\, p' \in \hat{V}_p \cap \mathcal{B}_k. \quad (4)$$

Now we are able to define the manifold V_p which occurs in the definition of the Kuranishi structure. For $p \in \mathcal{B}$ and for $\xi \in T_p \mathcal{B}$ small enough we define

$$\exp_p \xi \in \mathcal{B}$$

as the pointwise exponential map with respect to the metric $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$. Note that if $p \in \mathcal{B}_k$ and $\xi \in T_p \mathcal{B}_k$ then $\exp_p \xi \in \mathcal{B}_k$. Now we choose V_p as an open neighbourhood of zero in $\ker(\Pi_p \circ D_p) \subset T_p \mathcal{B}$ which is invariant under $I_k^{T\mathcal{B}}$ if $p \in \mathcal{B}_k$ and which is so small that we are able to define

$$\mathrm{ev}_p^0 \colon V_p \to \hat{V}_p, \quad \mathrm{ev}_p^0 := \exp_p|_{V_p}$$

and be recursion for $\nu \in \mathbb{N}$

$$\mathrm{ev}_p^\nu \colon V_p \to \hat{V}_p, \quad \xi \mapsto \exp_{\mathrm{ev}_p^{\nu-1}\xi}\left(R_p^{\mathrm{ev}_p^{\nu-1}\xi} \circ \Pi_p^{\mathrm{ev}_p^{\nu-1}\xi} \circ \mathcal{F} \circ \mathrm{ev}_p^{\nu-1}\xi\right),$$

and finally

$$\mathrm{ev}_p \colon V_p \to \hat{V}_p, \quad \mathrm{ev}_p := \lim_{\nu\to\infty} \mathrm{ev}_p^\nu.$$

We now define for each $p \in \mathcal{M}$ the obstruction bundle $E_p \to V_p$ by

$$E_p := (\mathrm{ev}_p)^* \bigoplus_{q \in Q_p} \mathcal{P}_q^{\mathrm{ev}_p} \hat{E}_q,$$

and the section $s_p \colon V_p \to E_p$ by

$$s_p := (\mathrm{ev}_p)^* \mathcal{F}.$$

The homeomorphisms ψ_p from $s_p^{-1}(0)$ to a neighbourhood of $p \in \mathcal{M}$ are defined by

$$\psi_p := \mathrm{ev}_p|_{s_p^{-1}(0)}.$$

Perhaps after shrinking the manifolds V_p we may assume using the assumption that the set $\hat{U}_q$ are closed for every $q \in Q$ that

$$Q_{\psi_p(x)} \subset Q_p, \quad \forall\, p \in \mathcal{M},\ \forall\, x \in s_p^{-1}(0). \tag{5}$$

This implies that for $p \in \mathcal{M}$ and $x \in s_p^{-1}(0)$ the set

$$V_{p\psi_p(x)} := \mathrm{ev}_{\psi_p(x)}^{-1}\Big(\mathrm{ev}_{\psi_p(x)}(V_{\psi_p(x)}) \cap \mathrm{ev}_p(V_p)\Big)$$

is open in $V_{\psi_p(x)}$. We now define

$$\phi_{p\psi_p(x)} \colon V_{p\psi_p(x)} \to V_p, \quad \phi_{p\psi_p(x)} := \mathrm{ev}_p^{-1} \circ \mathrm{ev}_{\psi_p(x)}.$$

Using again (5) we observe that for every $y \in V_{p\psi_p(x)}$ we have

$$\bigoplus_{q \in Q_{\psi_p(x)}} \mathcal{P}_q^{\mathrm{ev}_{\psi_p(x)}(y)} \hat{E}_q \subset \bigoplus_{q \in Q_p} \mathcal{P}_q^{\mathrm{ev}_p \circ \phi_{p\psi_p(x)}(y)} \hat{E}_q.$$

We now define the bundle maps $\hat{\phi}_{p\psi_p(x)} \colon E_{\psi_p(x)}|_{V_{p\psi_p(x)}} \to E_p$ as the map induced by the above inclusion. To define the isomorphisms $\Phi_{p\psi_p(x)} \colon N_{V_p} V_{\psi_p(x)} \to E_p / E_{\psi_p(x)}$ which are required for the tangent bundle of the Kuranishi structure, we observe that there is a natural identification of the normal bundle of $\mathrm{ev}_{\psi_p(x)}(V_{p\psi_p(x)})$ in $\mathrm{ev}_p(V_p)$

$$N_{\mathrm{ev}_p(V_p)} \mathrm{ev}_{\psi_p(x)}(V_{p\psi_p(x)}) \cong \bigoplus_{q \in Q_p \setminus Q_{\psi_p(x)}} \mathcal{P}_q^{\mathrm{ev}_{\psi_p(x)}} \hat{E}_q$$

due to the fact that

$$\Pi_{\psi_p(x)}^{\mathrm{ev}_{\psi_p(x)}(y)} \circ D_{\mathrm{ev}_{\psi_p(x)}(y)} \colon T_{\mathrm{ev}_{\psi_p(x)}(y)}\mathcal{B} \to \mathcal{E}_{\mathrm{ev}_{\psi_p(x)}(y)} / \bigoplus_{q \in Q_{\psi_p(x)}} \mathcal{P}_q^{\mathrm{ev}_{\psi_p(x)}(y)} \hat{E}_q$$

is already surjective. The isomorphisms $\Phi_{p\psi_p(x)}$ are defined to be the induced isomorphisms of the identification above. Finally, using (4), the involutions $I_k^{\mathcal{E}}$ induce involutions $\hat{I}_{p,k}$ on the obstruction bundle E_p for $p \in \mathcal{M}_k$. Hence we have proved that $\mathcal{M}$ admits a Kuranishi structure.

We next show in the integrable case the Kuranishi structure is of Arnold-Givental type. As it was explained in Remark 3.3, if the almost complex structure is integrable the operator D_u will interchange the involutions, i.e. for $k \in \mathbb{N}$ and $u \in \mathcal{F}^{-1}(0) \cap \mathcal{B}_k$ it holds that

$$I_k^{\mathcal{E}} \circ D_u = D_u \circ I_k^{T\mathcal{B}}. \tag{6}$$

If $p \in \mathcal{M}_k$ and $x \in V_{p,j}$ for $j \leq k$ then

$$T_x V_{p,j} = \mathrm{ev}_p^* \Big(\ker\big(\Pi_p^{\mathrm{ev}_p(x)} \circ D_{\mathrm{ev}_p(x)}\big) \cap \bigcap_{i=1}^{j-1} \big(\ker I_i^{T\mathcal{B}} - \mathrm{id}|_{T\mathcal{B}}\big)\Big),$$

and it follows from (6) and the invariance of the obstruction bundle under the involutions, that $I_i^{T\mathcal{B}}$ for $1 \leq i \leq j$ induce involutions on $TV_{p,j}$. Using these involutions one can endow the normal bundle with the structure of a bundle system of Arnold-Givental type.

It remains to treat the non-integrable case. In general (6) does not hold but by assertion (ii) of Theorem 3.1 we can homotop D_u through Fredholm operators to a Fredholm operator D_u^1 which interchanges the involutions by setting

$$D_u^\lambda := D_u - \lambda Q_u, \quad \lambda \in [0,1].$$

Now choose $\hat{V}_p^1 \subset U_p$ for every $p \in \mathcal{B}$, a finite set $Q^1 \in \mathcal{B}$, and for every $q \in Q^1$ a finite dimensional subspace $\hat{E}_q^1 \subset \mathcal{E}_q$ consisting of smooth sections, which satisfy again assertions (i) to (iii) but assertion (iv) replaces by

(iv)' For $p \in \mathcal{M}$ and $p' \in \hat{V}_p$ the operator

$$\Pi_p^{p',1} \circ D_{p'}^1 \colon T_{p'}\mathcal{B} \to \mathcal{E}_{p'} / \bigoplus_{q \in Q_p^1} \mathcal{P}_q^{p'} \hat{E}_q^1$$

is surjective.

Define for $p \in \mathcal{M}_k$ and $x \in V_{p,j}$ for $j \leq k$

$$W_{p,j} := \mathrm{ev}_p^* \Big(\ker\big(\Pi_p^{\mathrm{ev}_p(x),1} \circ D_{\mathrm{ev}_p(x)}^1\big) \cap \bigcap_{i=1}^{j-1} \big(\ker I_i^{T\mathcal{B}} - \mathrm{id}|_{T\mathcal{B}}\big)\Big).$$

Note that $W_{p,j}$ is invariant under the involutions $I_i^{T\mathcal{B}}$ for $1 \leq i \leq j$. One can now define a bundle system of Arnold-Givental type N^1X over $\bigcup_{j=2}^m X_j$ where $F_{1,p}$ is given by $\bigcup_{j=2}^m W_{p,j}|_{V_{p,j+1}}/W_{p,j+1}$, $F_{2,p} = \bigcup_{j=2}^m E_{p,j}|_{V_{p,j+1}}/E_{p,j+1}$, and the transition functions are defined in a similar manner as in the case of the normal bundle. Using the homotopy between D_u and D_u^1 one shows that

$$[N^1X] = [NX] \in \bigoplus_{j=2}^m KO(X_j).$$

It follows that the Kuranishi structure is of stable Arnold-Givental type. This proves the Theorem. □

4 Moment Floer homology

Moment Floer homology was introduces in [Fr2]. Moment Floer homology is a tool to count intersection points of some Lagrangians in Marsden-Weinstein quotients which are fixpoint set of some antisymplectic involution. In general, due to the bubbling phenomenon, the ordinary Floer homology for Lagrangians in Marsden-Weinstein quotients cannot be defined by standard means, see [Ch, ChOh] for a computation of the Floer homology of Lagrangian torus fibers of Fano toric manifolds. To overcome the bubbling problem one replaces Floer's equations by the symplectic vortex equations to define the boundary operator. Under some topological assumptions on the enveloping manifold one can prove compactness of the relevant moduli spaces of the symplectic vortex equations. In the special case where the two Lagrangians are hamiltonian isotopic to each other one can use the antisymplectic involution to prove that moment Floer homology is equal to the singular homology of the Lagrangian with coefficients in some Novikov ring. This leads to a prove of the Arnold-Givental conjecture for some class of Lagrangians in Marsden-Weinstein quotients which are fixpoint sets of some antisymplectic involution. We will give in this section proofs of the main properties of the symplectic vortex equations to define moment Floer homology and refer to [Fr2] for complete details. To compute it we will need the techniques of section 3. These techniques were not available in [Fr2] and hence moment Floer homology could there only be computed under some additional monotonicity assumption which is removed here.

4.1 The set-up

In this subsection we introduce the notation to define the symplectic vortex equations and formulate the hypotheses under which compactness of the relevant moduli spaces can be proven.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ which acts covariantly on a manifold M, i.e. there exists a smooth homomorphism $\psi : G \to \mathrm{Diff}(M)$. We will often drop ψ and identify g with $\psi(g)$.

For $\xi \in \mathfrak{g}$ we denote by X_ξ the vector field $M \to TM$ which is generated by the one-parameter subgroup generated by ξ, i.e.

$$X_\xi(x) := \frac{d}{dt}\Big|_{t=0} \exp(t\xi)(x), \quad \forall\, x \in M.$$

We shall use the linear mapping $L_x : \mathfrak{g} \to T_xM$ defined by

$$L_x\xi := X_\xi(x) \in T_xM.$$

We will denote the adjoint action of G on $\mathfrak{g}$ by

$$g\xi g^{-1} = \mathrm{Ad}(g)\xi = \frac{d}{dt}\Big|_{t=0} g\exp(t\xi)g^{-1}.$$

If I is an open intervall, $t_0 \in I$, and $g : I \to M$ is a smooth path, we will write

$$(g^{-1}\partial_t g)(t_0) := d\mathcal{L}^{-1}_{g(t_0)}(g(t_0))\partial_t g(t_0) \in T_{\mathrm{id}}G = \mathfrak{g}$$

where $\mathcal{L}_g \in \mathrm{Diff}(G)$ is the left-multiplication by g.

Assume that the Lie algebra is endowed with an inner product $\langle\cdot,\cdot\rangle$ which is invariant under the adjoint action of the Lie group. If (M,ω) is symplectic, we say that the action of G is Hamiltonian, if there exists a moment map for the action, i.e. an equivariant function $\mu : M \to \mathfrak{g}$ [3] where the action of G on $\mathfrak{g}$ is the adjoint action, such that for every $\xi \in \mathfrak{g}$

$$d\langle\mu(\cdot),\xi\rangle = \iota(X_\xi)\omega.$$

Note that the function $\langle\mu(\cdot),\xi\rangle$ is a Hamiltonian function for the vector field X_ξ. Observe that if $\xi \in Z(\mathfrak{g})$ [4], then $\mu_\xi(\cdot) := \mu(\cdot) + \xi$ is also a moment map for the action of G. Hence the moment map is determined by the action up to addition of a central element in each connected component of M.

We assume now that (M,ω) is a symplectic (not necessarily compact) connected manifold, and G a compact connected Lie group that acts on M by Hamiltonian symplectomorphisms as above. We assume that the action is effective, i.e. the homomorphism ψ is injective. Let L_0 and L_1 be two closed Lagrangian submanifolds of M. We do not require that the Lagrangians are G-invariant but we assume throughout this section the following compatibility condition with G.

(H1) *For $j \in \{0,1\}$ there exist antisymplectic involutions $R_j \in \mathrm{Diff}(M)$, i.e.*

$$R_j^*\omega = -\omega, \quad R_j^2 = \mathrm{id},$$

which commute which G, i.e. for every $g \in G$ the symplectomorphism $R_j\psi(g)R_j$ lies in the image of ψ, such that

$$L_j = \mathrm{Fix}(R_j) = \{x \in M : R_j(x) = x\}.$$

[3]Some authors use the convention that the moment map takes values in the dual of the Lie algebra. Since we have an inner product we can identify the Lie algebra with its dual.

[4]The centraliser $Z(\mathfrak{g})$ consists of all $\xi \in \mathfrak{g}$ such that $[\xi,\eta] = 0$ for every $\eta \in \mathfrak{g}$

The maps R_j lead to Lie group Automorphisms $S_j : G \to G$ defined by

$$S_j(g) := \psi^{-1}(R_j \psi(g) R_j), \quad \forall\, g \in G. \tag{7}$$

Note that S_j are involutative, i.e.

$$S_j^2 = \mathrm{id}.$$

We assume that the inner product in the Lie algebra is also invariant under the differentials of S_j at the identity. These are determined by the formula

$$X_{dS_j(\mathrm{id})(\xi)}(x) = dR_j(R_j(x))^{-1} X_\xi(R_j x), \quad \forall\, x \in M.$$

In the following we will write $\dot{S}_j$ for $dS_j(\mathrm{id})$. If one identifies G with $\psi(G)$, then formally

$$\dot{S}_j = \mathrm{Ad}(R_j).$$

Let μ be a moment map for the action of G on M. We further impose the following hypothesis throughout this section.

(H2) *The moment map μ is proper, zero is a regular value of μ, and G acts freely on $\mu^{-1}(0)$, i.e. $\psi(g)p = p$ for $p \in \mu^{-1}(0)$ implies that $g = \mathrm{id}$.*

The Marsden-Weinstein quotient is defined to be the set of G-orbits in $\mu^{-1}(0)$

$$\bar{M} := M//G := \mu^{-1}(0)/G,$$

i.e. $x, y \in \mu^{-1}(0)$ are equivalent if there exists $g \in G$ such that $\psi(g)x = y$. It follows from hypothesis (H2) that $\bar{M}$ is a compact manifold of dimension

$$\dim(\bar{M}) = \dim(M) - 2\dim(G).$$

The Marsden-Weinstein quotient carries a natural symplectic structure induced from the symplectic structure on M, see [MS1, Proposition 5.40].

We denote by

$$G_{L_j} := \{g \in G : gL_j = L_j\}$$

for $j \in \{0, 1\}$ the isotropy subgroup of the Lagrangian L_j. It follows directly from the definitions that $G_{S_j} := \{g \in G : S_j g = g\}$ is a subgroup of G_{L_j}. If $\mu^{-1}(0) \cap L_j \neq \emptyset$, than the two groups agree. To see that, note that by (H1) there exists $p \in L_j$ whose isotropy subgroup is trivial, i.e. $G_p := \{g \in G : gp = p\} = \{\mathrm{id}\}$. If $g \in G_{L_j}$ then $g^{-1} S_j(g) p = p$ and hence $g \in G_{S_j}$.

We denote by $\mathfrak{g}_{L_j}$ the Lie-algebra of G_{L_j}. Note that if $\mu^{-1}(0) \cap L_j \neq \emptyset$

$$\mathfrak{g}_{L_j} = \{\xi \in \mathfrak{g} : \dot{S}_j(\xi) = \xi\}, \quad \mathfrak{g}_{L_j}^{\perp} = \{\xi \in \mathfrak{g} : \dot{S}_j(\xi) = -\xi\},$$

where $\perp$ stands for the invariant inner product defined above. The following proposition says that the two Lagrangians induce Lagrangian submanifolds in the Marsden-Weinstein quotient.

Proposition 4.1 *Assume* $(H1)$ *and* $(H2)$*. Then the subsets of* $\bar{M}$

$$\bar{L}_j := G(L_j \cap \mu^{-1}(0))/G$$

are Lagrangian submanifolds of $\bar{M}$ *and they are naturally diffeomorphic to*

$$(L_j \cap \mu^{-1}(0))/G_{L_j}.$$

The following example shows that Proposition 4.1 will in general be wrong if we do not assume hypothesis $(H1)$.

Example 4.2 *Consider the standard action of* S^1 *on* $\mathbb{C}^2$ *given by*

$$(z_1, z_2) \mapsto (e^{i\theta} z_1 e^{i\theta} z_2).$$

A moment map for this action is given by

$$\mu(z) := \frac{i}{2}(|z|^2 - 1)$$

and the Marsden-Weinstein quotient is the two sphere. Consider now tthe family of Lagrangian submanifolds of $\mathbb{C}^2$ *given by*

$$L_a := \{(x_1 + ia, x_2) : x_1, x_2 \in \mathbb{R}\}$$

where $a \in \mathbb{R}$*. Note that if* $a = 0$ *then* L_a *equals the fixpoint set of the antisymplectic involution on* $\mathbb{C}^2$ *which is given by complex conjugation. This involution commutes with the* S^1*-action. For* $a \neq 0$ *the Lagrangians* L_a *do not satisfy hypothesis* $(H1)$*. Consider the chart*

$$\left\{ \frac{z_2}{z_1} : |z_1|^2 + |z_2|^2 = 1,\ z_2 \neq 0 \right\} \cong \mathbb{C}$$

of $\mu^{-1}/S^1 \cong S^2$*. Then the images of* $\bar{L}_a$ *are given by*

$$\left\{ \frac{t\sqrt{1 - t^2 - a^2}}{\sqrt{1 - t^2}} \pm \frac{iat}{\sqrt{1 - t^2}} : |t| \leq 1 - a^2 \right\}.$$

For $a \neq 0$ *these are figure eights with nodal point* $(0, 0)$*.*

To prove Proposition 4.1 we need two lemmas.

Lemma 4.3 *Assume* $(H1)$ *and* $(H2)$*. The Lagrangians* L_j *intersect cleanly with* $\mu^{-1}(0)$*, i.e.* $\mu^{-1}(0) \cap L_j$ *is a submanifold of* $\mu^{-1}(0)$ *and for every* $p \in \mu^{-1}(0) \cap L_j$ *we have* $T_p\mu^{-1}(0) \cap T_pL_j = T_p(\mu^{-1}(0) \cap L_j)$*.*

Proof: We may assume without loss of generality that $\mu^{-1}(0) \cap L_j \neq \emptyset$. For $p \in \mu^{-1}(0) \cap L_j$ we claim that

$$d\mu(p)T_pL_j = \mathfrak{g}_{L_j}^{\perp}. \tag{8}$$

We first calculate for $v \in T_pL_j$ and $\xi \in \mathfrak{g}_{L_j}$

$$\langle d\mu(p)v, \xi\rangle = d\langle \mu, \xi\rangle(p)v = \omega(X_\xi(p), v) = 0$$

and hence

$$d\mu(p)T_pL_j \subset \mathfrak{g}_{L_j}^\perp.$$

To prove that equality holds in (8) it suffices to show the following implication

$$\xi \in \mathfrak{g}_{L_j}^\perp,\ \langle \xi, d\mu(p)v\rangle = 0\ \forall v \in T_pL_j \quad \Longrightarrow \quad \xi = 0. \tag{9}$$

Assume that ξ satisfies the assumption in (9). Let $w \in T_pM$. Then $w = w_1 + w_2$, where $w_1 \in T_pL_j$, i.e. $dR_jw_1 = w_1$, and $dR_jw_2 = -w_2$. We calculate

$$\begin{aligned}
\langle \xi, d\mu(p)w\rangle &= \langle \xi d\mu(p)w_2\rangle \\
&= \omega(X_\xi(p), w_2) \\
&= \omega(X_\xi(p), -dR_jw_2) \\
&= \omega(dR_jX_\xi(p), w_2) \\
&= \omega(X_{\dot{S}_j\xi}(p), w_2) \\
&= -\omega(X_\xi(p), w_2) \\
&= -\langle \xi, d\mu(p)w\rangle
\end{aligned}$$

and hence

$$\langle \xi, d\mu(p)w\rangle = 0\ \forall w \in T_pM.$$

Since $0 = \mu(p)$ is a regular value of μ it follows that $d\mu(p)$ is surjective and hence $\xi = 0$. This proves (9) and hence (8). If U is a sufficiently small open neighbourhood of 0 in $\mathfrak{g}_{L_j}^\perp$, then $\mu^{-1}(U)$ is a submanifold of M. It follows from (8), that $\mu^{-1}(0)$ and $L_j \cap \mu^{-1}(U)$ intersect transversally in $\mu^{-1}(U)$. Hence $\mu^{-1}(0)$ and L_j intersect cleanly. $\square$

Lemma 4.4 *Assume* $(H1)$ *and* $(H2)$*. If* $p \in L_j$, $G_p = \{g \in G : gp = p\} = \{\mathrm{id}\}$ *and* $\psi(g)p \in L_j$ *for some* $g \in G$*, then* $g \in G_{L_j}$*.*

Proof: Because $L_j = \mathrm{Fix}(R_j)$,

$$R_jgR_jp = gp.$$

Since $R_jG = GR_j$ there exists $\tilde{g} \in G$ such that

$$\tilde{g}x = R_jgR_jx \quad \forall\ x \in M.$$

Hence

$$(\tilde{g})^{-1}gp = p$$

and because $G_p = \{\mathrm{id}\}$

$$\tilde{g} = g.$$

Hence $g = R_j g R_j$ and for every $q \in L_j$

$$gq = R_j gq.$$

This implies that $gq \in L_j$ and hence $g \in G_{L_j}$. $\square$

Proof of Proposition 4.1: It follows from Lemma 4.3 and the fact that G_{L_j} acts freely on $L_j \cap \mu^{-1}(0)$ that $(L_j \cap \mu^{-1}(0))/G_{L_j}$ is a manifold. There is an obvious surjective map from $(L_j \cap \mu^{-1}(0))/G_{L_j}$ to $\bar{L}_j$ which assigns to a representative $x \in L_j \cap \mu^{-1}(0)$ of an equivalence class in $(L_j \cap \mu^{-1}(0))/G_{L_j}$ the equivalence class of x in $\bar{L}_j$. It follows from Lemma 4.4 that this map is an injection. $\square$

In addition we make the following topological assumptions.

(H3) *$\pi_2(M)$, $\pi_1(M)$, $\pi_1(L_j)$, and $\pi_0(L_j)$ for $j \in \{0,1\}$ are trivial.*[5]

Convex structures for Hamiltonian group actions on symplectic manifolds were introduced in [CGMS]. We give a similar definition which takes care of the Lagrangian submanifolds. Recall that an almost complex structure is called ω-compatible if

$$\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$$

is a Riemannian metric on TM. We say that an almost complex structure is G-invariant, if

$$J(z) = g_* J(z) := d\psi(g)^{-1}(gz) J(gz) d\psi(g) z, \quad \forall\, g \in G,\ \forall\, z \in M.$$

It follows from [MS1, Proposition 2.50] that the space of G-invariant compatible almost complex structures is nonempty and contractible.

Definition 4.5 *A* **convex structure** *on $(M, \omega, \mu, L_0, L_1)$ is a pair (f, J) where J is a G-invariant ω-compatible almost complex structure on M which satisfies*

$$dR_j(R_j z) J(R_j z) dR_j(z) = -J(z), \quad \forall z \in M. \tag{10}$$

for $j \in \{0,1\}$ and $f : M \to [0, \infty)$ is a smooth function satisfying the following conditions.

(C1) *f is G-invariant and proper.*

[5] Most of the results of this paper could be generalized to the case, where we replace (H3) by the following weaker assumption $(H3')$

(H3') *For every smooth map $v : (B, \partial B) \to (M, L_j)$, we have*

$$\int_B v^* \omega = 0.$$

However, if we only assume (H3') instead of (H3), then our path space will in general neither be connected nor simply connected. In particular, there will be no well defined action functional.

(C2) *There exists a constant $c_0 > 0$ such that*

$$f(x) \geq c_0 \quad \Longrightarrow \quad \langle \nabla_\xi \nabla f(x), \xi \rangle \geq 0,\ df(x)J(x)X_{\mu(x)}(x) \geq 0,\ \mu(x) \neq 0$$

for every $x \in M$ and every $\xi \in T_xM$. Here ∇ denotes the Levi-Civita connection of the metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$.

(C3) *For every $p \in L_j$ it holds that $\nabla f(p) \in T_pL_j$.*

As our fourth hypothesis we assume that a convex structure exists

(H4) *There exists a convex structure (f, J_0) on $(M, \omega, \mu, L_0, L_1)$.*

A convex structure will guarantee, that solutions of our gradient equation will remain in a compact domain.

The main examples we have in mind, are of the following form. The symplectic manifold (M, ω) equals the complex vector space $\mathbb{C}^n$ endowed with its canonical symplectic structure ω_0, the Lagrangians equal some linear Lagrangian subspace of $\mathbb{C}^n$, and the group action ψ is given by some injective linear representation of a connected compact Lie group G to $U(n)$.

For a linear Lagrangian subspace L of $\mathbb{C}^n$ there is a $\mathbb{R}$-linear splitting

$$\mathbb{C}^n = L \oplus J_0 L$$

where J_0 is the standard complex structure given by multiplication with i. Let $R = R_L$ be the canonical antisymplectic involution given by

$$R(x + J_0 y) = x - J_0 y$$

for $x, y \in L$. Hypothesis (H1) means

$$\rho(G)R = R\rho(G).$$

In the special case where $\rho = \mathrm{id}$ and $L = \mathbb{R}^n$, the induced involution S on G is given by complex conjugation. To see that, choose $A \in U(n)$ and $z \in \mathbb{C}$. We calculate

$$S(A)z = RAR(z) = RA\bar{z} = \bar{A}z$$

and hence $S(A) = \bar{A}$.

The Lie algebra $u(n)$ of $U(n)$ carries a natural invariant inner product given by

$$\langle A, B \rangle := \mathrm{trace}(A^* B),$$

where A^* is the complex conjugated transposed of A. Let $\dot{\rho} : \mathfrak{g} \to u(n)$ be the induced representation of ρ. Endow $\mathfrak{g}$ with the invariant inner product

$$\langle \xi_1, \xi_2 \rangle_\rho = \langle \dot{\rho}(\xi_1), \dot{\rho}(\xi_2) \rangle, \quad \forall\ \xi_1, \xi_2 \in \mathfrak{g}.$$

Let $\dot{\rho}^* : u(n) \to \mathfrak{g}$ be the adjoint of $\dot{\rho}$, i.e.

$$\langle \dot{\rho}(\xi), \eta \rangle = \langle \xi, \dot{\rho}^*(\eta) \rangle_\rho \quad \forall\ \xi \in \mathfrak{g}, \eta \in u(n),$$

then a moment map μ for the action of G is given by

$$\mu(z) = -\frac{1}{2}\dot{\rho}^*(izz^*) - \tau$$

where τ is a central element of $\mathfrak{g}$. Since $\dot{\rho}$ is isometric with respect to our inner products, $\langle \cdot, \cdot \rangle$ is $\dot{S}$ invariant.

A convex structure on $(\mathbb{C}^n, \omega_0, \mu, L)$ is defined for example by

$$(f, J) = (\frac{1}{2}|z|^2, J_0).$$

Example 4.6 (Toric manifolds) *Let A be a $k \times n$-matrix of rank k whose entries are positive integers. Let the k-torus T^k act on $\mathbb{C}^n$ by*

$$z \mapsto \exp(2\pi i A\theta^T)z$$

where $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}/\mathbb{Z} \times \ldots \times \mathbb{R}/\mathbb{Z} = T^k$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. For some $\tau \in i\mathbb{R}^k$ which is equal to the Lie algebra of the torus a moment map for the torus action above is defined by

$$\mu(z) = \frac{1}{2i}(A^T A)^{-1} A^T \begin{pmatrix} |z_1|^2 \\ \vdots \\ |z_n|^2 \end{pmatrix} - \tau.$$

If T^k acts freely on $\mu^{-1}(0)$, then the Marsden-Weinstein quotient $\mathbb{C}^n//T^k = \mu^{-1}(0)/T^k$ is called a toric manifold.

Example 4.7 (Grassmannians) *There is a natural action of the unitary group $U(k)$ on $\mathbb{C}^{n\times k}$, the space of k-frames in $\mathbb{C}^n$, having the moment map*

$$\mu(B) = \frac{1}{2i}(B^*B - id).$$

The symplectic quotient is the complex Grassmannian manifold $G_{\mathbb{C}}(n,k)$. Let $L = \mathbb{R}^{n\times k}$ be the space of real k-frames. Its isotropy subgroup is the orthogonal group $O(k)$ and the induced Lagrangian in the symplectic quotient equals the real Grassmannian $G_{\mathbb{R}}(n,k)$.

Remark 4.8 (Naturality) *For $U \in U(n)$ let the representation ρ_U be given by*

$$\rho_U(g) = U\rho(g)U^{-1}, \quad \forall g \in G.$$

Then a moment map for this action is given by

$$\mu_U(z) = \mu(U^{-1}z)$$

and there is a natural induced isomorphism from $\bar{M}$ to $\bar{M}_U := \mu_U^{-1}(0)/G$ given by

$$[z] \mapsto [Uz].$$

Defining the linear Lagrangian subspace

$$L_U := U(L)$$

the image of $\bar{L}$ *under the above isomorphism equals*

$$\bar{L}_U := G(\mu_U^{-1}(0) \cap L_U)/G.$$

Because the group $U(n)$ *acts transitively on the set of Lagrangian subspaces of* $\mathbb{C}^n$ *one can always assume after applying some* U *as above, that* $L = \mathbb{R}^n$.

4.2 The symplectic vortex equations on the strip

In this subsection we show how one can derive the symplectic vortex equations from an action functional.

We define the path space $\mathscr{P}$ by

$$\mathscr{P} := \{(x, \eta) \in C^\infty([0,1], M \times \mathfrak{g}) : x(j) \in L_j,\ \eta(j) \in \mathfrak{g}_{L_j}^\perp,\ j \in \{0,1\}\}, \tag{11}$$

The assumption (H3) implies that $\mathscr{P}$ is connected and simply connected. The gauge group $\mathcal{H}$ is defined by

$$\mathcal{H} := \{g \in C^\infty([0,1], G) : g(j) \in G_{L_j},\ g(j)^{-1}\partial_t g(j) \in \mathfrak{g}_{L_j}^\perp,\ j \in \{0,1\}\}.$$

The group structure is the pointwise multiplication of G. The gauge group $\mathcal{H}$ acts on $\mathscr{P}$ as follows

$$g_*(x, \eta) = (gx, g\eta g^{-1} - \partial_t g g^{-1}), \quad g \in \mathcal{H}.$$

Choose a path $x_0 : [0,1] \to M$ with $x_0(j) \in L_j$ for $j \in \{0,1\}$. For a smooth family of G-invariant functions $H_t : M \to \mathbb{R}$ for $t \in [0,1]$, we define the action functional

$$\mathcal{A}_{\mu,H} : \mathscr{P} \to \mathbb{R}$$

by

$$\mathcal{A}_{\mu,H}(x,\eta) = -\int_{[0,1]\times[0,1]} \bar{x}^*\omega + \int_0^1 \big(\langle \mu(x(t)), \eta(t)\rangle - H_t(x(t))\big)\,dt,$$

where $\bar{x} : [0,1] \times [0,1] \to M$ is a smooth map, which satisfies

$$\bar{x}(t,1) = x(t), \quad \bar{x}(t,0) = x_0(t), \quad \bar{x}(0,s) \in L_0, \quad \bar{x}(1,s) \in L_1.$$

Since ω is closed and vanishes on the Lagrangians the assumption that $\pi_2(M) = 0$ and $\pi_1(L_j) = 0$ for $j \in \{0,1\}$ together with Stokes theorem implies that the value of $\mathcal{A}_{\mu,H}(x,\eta)$ does not depend on the choice of $\bar{x}$. Moreover, $\mathcal{A}_{\mu,H}$ is invariant under the action of $\mathcal{H}_0$, the path-connected component of the identity of $\mathcal{H}$. To see this, let $g \in \mathcal{H}_0$, then there exists $h : [0,1] \times [0,1] \to G$ with

$$h(t,1) = g(t), \quad h(t,0) = \mathrm{id}, \quad h(j,s) \in G_{L_j}, \quad h^{-1}\partial_t h(j,s) \in \mathfrak{g}_{L_j}^\perp,\ j \in \{0,1\}.$$

The claim follows with $\overline{g^*x} = h\bar{x}$.

The tangent space $T_{(x,\eta)}\mathscr{P}$ of the path space $\mathscr{P}$ at $(x,\eta) \in \mathscr{P}$ is defined as the vector space

$$\{(\hat{x},\hat{\eta}) \in C^\infty([0,1], x^*TM \times \mathfrak{g}) : \hat{x}(j) \in T_{x(j)}L_j,\ \hat{\eta}(j) \in \mathfrak{g}_{L_j}^\perp,\ j \in \{0,1\}\}.$$

A family of G-invariant, ω-compatible, almost complex structures J_t determines an inner product on $\mathscr{P}$ by

$$\langle(\hat{x}_1,\hat{\eta}_1),(\hat{x}_2,\hat{\eta}_2)\rangle = \int_0^1 (\langle \hat{x}_1(t), \hat{x}_2(t)\rangle_t + \langle \hat{\eta}_1(t), \hat{\eta}_2(t)\rangle) dt \tag{12}$$

for $(\hat{x}_1,\hat{\eta}_1),(\hat{x}_2,\hat{\eta}_2) \in T_{(x,\eta)}\mathscr{P}$, where

$$\langle\cdot,\cdot\rangle_t = \langle\cdot,\cdot\rangle_{J_t} = \omega(\cdot, J_t\cdot).$$

The gradient of $\mathcal{A}_{\mu,H}$ with respect to the above inner product as usual defined by

$$d\mathcal{A}_{\mu,H}(x,\eta)[\hat{x},\hat{\eta}] = \langle \mathrm{grad}\mathcal{A}_{\mu,H}(x,\eta), [\hat{x},\hat{\eta}]\rangle$$

is given by

$$\mathrm{grad}\mathcal{A}_{\mu,H}(x,\eta) = \begin{pmatrix} J_t(\dot{x} + X_\eta(x) - X_{H_t}(x)) \\ \mu(x) \end{pmatrix}.$$

The set $\mathrm{crit}(\mathcal{A}) \subset \mathscr{P}$ of critical points of $\mathcal{A}_{\mu,H}$ consists of paths $(x,\eta) : [0,1] \to M \times \mathfrak{g}$ which satisfy

$$\dot{x} + X_\eta(x) = X_{H_t}(x), \quad \mu(x) = 0, \quad x(j) \in L_j, \quad \eta(j) \in \mathfrak{g}_{L_j}^\perp,\ j \in \{0,1\}.$$

Since G acts freely on $\mu^{-1}(0)$ the group $\mathcal{H}$ acts freely on $\mathrm{crit}(\mathcal{A})$. If $\bar{H}$ is the induced Hamiltonian function of H in the Marsden-Weinstein quotient $\bar{M}$ and $\phi^t_{\bar{H}}$ its flow, i.e.

$$\frac{d}{dt}\phi^t_{\bar{H}} = X_{\bar{H}_t} \circ \phi^t_{\bar{H}}, \quad \phi^0_{\bar{H}} = \mathrm{id},$$

then we will prove in Lemma 4.11 below that there is a natural bijection

$$\mathrm{crit}(\mathcal{A})/\mathcal{H} \cong \phi^1_{\bar{H}}(\bar{L}_0) \cap \bar{L}_1.$$

Let

$$\Theta = \{z = s + it \in \mathbb{C} : 0 \le t \le 1\}$$

be the strip. The flow lines of the vector field $\mathrm{grad}\mathcal{A}_{\mu,H}$ are pairs $(u,\Psi) \in C^\infty_{loc}(\Theta, M \times \mathfrak{g})$, which satisfy the following partial differential equation

$$\begin{gathered} \partial_s u + J_t(u)(\partial_t u + X_\Psi(u) - X_{H_t}(u)) = 0 \\ \partial_s \Psi + \mu(u) = 0 \\ u(s,j) \in L_j, \quad \eta(s,j) \in \mathfrak{g}_{L_j}^\perp,\ \ j \in \{0,1\}. \end{gathered} \tag{13}$$

We define further the gauge group

$$\mathcal{G}_{loc} = \{g \in C^\infty_{loc}(\Theta, G) : g(s,j) \in G_{L_j},\, g^{-1}\partial_t g(s,j) \in \mathfrak{g}_{L_j}^\perp,\, j \in \{0,1\}\}.$$

Solutions of the problem (13) are invariant under the action of $\mathcal{H}$ but not of $\mathcal{G}_{loc}$. To make the problem invariant under the gauge group $\mathcal{G}_{loc}$, we introduce an additional variable Φ. Given a solution (u_0, Ψ_0) of (13) and $g \in \mathcal{G}_{loc}$ then $(u, \Psi, \Phi) = (gu_0, g\Psi_0 g^{-1} - g^{-1}\partial_t g, -g^{-1}\partial_s g)$ is a solution of the so called **symplectic vortex equations on the strip**

$$\begin{gathered} \partial_s u + X_\Phi(u) + J_t(u)(\partial_t u + X_\Psi(u) - X_{H_t}(u)) = 0 \\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \mu(u) = 0 \\ u(s,j) \in L_j, \quad \Phi(s,j) \in \mathfrak{g}_{L_j}, \quad \Psi(s,j) \in \mathfrak{g}_{L_j}^\perp \quad j \in \{0,1\}. \end{gathered} \tag{14}$$

Moreover, (14) is invariant under the action of $g \in \mathcal{G}_{loc}$ given by

$$g_*(u, \Psi, \Phi) = (gu, g\Psi g^{-1} - \partial_t g g^{-1}, g\Phi g^{-1} - \partial_s g g^{-1}).$$

On the other hand each solution of (14) is gauge equivalent to a solution of (13). To see that, let (u, Ψ, Φ) be a solution of (14) and take the solution $g : \Theta \to G$ of the following ordinary differential equation on the strip

$$\partial_s g = g\Phi, \quad g(0,t) = \mathrm{id}.$$

Then $g \in \mathcal{G}_{loc}$ and $g_*\Phi = 0$. In the terminology of gauge theory, this means that solutions of (13) are solutions of (14) in so called **radial gauge**.

Remark 4.9 *If one introduces the connection $A = \Phi ds + \Psi dt$ on the trivial G-bundle over the strip, then the first two equations of (14) can be written as*

$$\bar\partial_{J,H,A}(u) = 0, \qquad *F_A + \mu(u) = 0.$$

These equations were discovered independently by D.Salamon and I.Mundet (see [CGS], [CGMS], and [Mu]). In the physics literature they are known as gauged sigma models.

Remark 4.10 (Naturality) *Solutions of the problem (14) have the following properties. Let K_t be some smooth family of G-invariant functions on M and let $\psi_K^t : M \to M$ be the Hamiltonian symplectomorphism defined by*

$$\frac{d}{dt}\psi_K^t = X_{K_t} \circ \psi_K^t, \quad \psi_K^0 = \mathrm{id}.$$

If (u, Ψ, Φ) is a solution of (14), then

$$(\tilde u, \tilde\Psi, \tilde\Phi)(s,t) := (\psi_K^{-t} \circ u, \Psi, \Phi)(s,t)$$

is also a solution of (14) with H, J, L_0, L_1 replaced by

$$\tilde H_t := (H_t - K_t) \circ \psi_K^t, \quad \tilde J_t := (\psi_K^t)^* J_t, \quad \tilde L_0 = L_0, \quad \tilde L_1 = \psi_K^{-1} L_1.$$

In particular, by choosing $H_t = K_t$ one can always assume that $H \equiv 0$.

Proposition 4.11 *There is a natural bijection between* $\mathrm{crit}(\mathcal{A})/\mathcal{H}$ *and* $\phi^1_{\bar{H}}(\bar{L}_0)\cap \bar{L}_1$.

Proof: By Remark 4.10 we may assume without loss of generality that $H = 0$. Denote by π the canonical projection from $\mu^{-1}(0)$ to $\bar{M} = \mu^{-1}(0)/G$. If $q \in \bar{L}_0 \cap \bar{L}_1$, then there exists $x_0 \in L_0$, $x_1 \in L_1$, and $h \in G$ such that

$$\pi(x_0) = \pi(x_1) = q, \quad x_1 = hx_0.$$

Choose a smooth path $g \in C^\infty([0,1], G)$ such that

$$g(0) = \mathrm{id}, \quad g(1) = h, \quad (\partial_t g)g^{-1}(0) \in \mathfrak{g}^\perp_{L_0}, \quad (\partial_t g)g^{-1}(1) \in \mathfrak{g}^\perp_{L_1}.$$

Such a path exists, since G is connected. Now define

$$I : \bar{L}_0 \cap \bar{L}_1 \to \mathrm{crit}(\mathcal{A})/\mathcal{H}, \quad q \mapsto [(g(t)x_0, -(\partial_t g)g^{-1}(t))].$$

Here $[\cdot,\cdot]$ denotes the equivalence class in $\mathrm{crit}(\mathcal{A})/\mathcal{H}$.
We have to show that I is well defined. To see that choose another quadruple $(\tilde{x}_0, \tilde{x}_1, \tilde{h}, \tilde{g})$ which satisfies the relations above. It follows from Lemma 4.4 that there exists $h_j \in G_{L_j}$ for $j \in \{0,1\}$ such that

$$\tilde{x}_j = h_j x_j.$$

Define

$$\gamma(t) := g(t)h_0^{-1}\tilde{g}(t)^{-1}.$$

Then

$$\gamma_*(g(t)x_0, -(\partial_t g)g^{-1}) = (\tilde{g}(t)\tilde{x}_0, -(\partial_t\tilde{g})\tilde{g}^{-1})$$

and

$$\gamma \in \mathcal{H}.$$

This shows that I is well defined. The verification that I is a bijection is easy, namely to construct the inverse of I map $(x,\eta) \in \mathrm{crit}(\mathcal{A})$ to $\pi(x)(0)$. $\square$

Remark 4.12 (Extension) *Every solution can be extended to the whole complex plane. To see that let* (u, Ψ, Φ) *be a solution of (14) and assume that*

$$J_j(z) = -dR_j(R_j z)J_j(R_j z)dR_j(z), \quad z \in M,\ j \in \{0,1\}. \tag{15}$$

For simplicity, assume also that $H \equiv 0$. *Let* $\hat{J}_t$ *for* $t \in \mathbb{R}$ *be the unique* G-*invariant extension of* ω-*compatible almost complex structures on* M *defined by the following conditions*

$$\begin{aligned} \hat{J}|_{[0,1]\times M} &= J \\ \hat{J}_{2n-t}(z) &= -dR_0(R_0 z)\hat{J}_{2n+t}(R_0 z)dR_0(z), \quad n \in \mathbb{Z},\ t \in (0,1] \\ \hat{J}_{2n+1-t}(z) &= -dR_1(R_1 z)\hat{J}_{2n+1+t}(R_1 z)dR_1(z), \quad n \in \mathbb{Z},\ t \in (0,1]. \end{aligned}$$

For $n \in \mathbb{Z}$ and $t \in (0,1]$ let $(\hat{u}, \hat{\Psi}, \hat{\Phi}) \in W^{1,p}_{loc}(\mathbb{C}, M \times \mathfrak{g} \times \mathfrak{g})$ be defined by the conditions

$$\begin{aligned}
(\hat{u}, \hat{\Psi}, \hat{\Phi})|_{\Theta} &= (u, \Psi, \Phi), \\
(\hat{u}, \hat{\Psi}, \hat{\Phi})(s, 2n - t) &= (R_0 \hat{u}, -\dot{S}_0(\hat{\Psi}), \dot{S}_0(\hat{\Phi}))(s, 2n + t), \\
(\hat{u}, \hat{\Psi}, \hat{\Phi})(s, 2n + 1 - t) &= (R_1 \hat{u}, -\dot{S}_1(\hat{\Psi}), \dot{S}_1(\hat{\Phi}))(s, 2n + 1 + t).
\end{aligned}$$

Here S_j for $j \in \{0,1\}$ was defined in (7). The map $(\hat{u}, \hat{\Psi}, \hat{\Phi})$ solves

$$\begin{gathered}
\partial_s \hat{u} + X_{\hat{\Phi}}(\hat{u}) + J_t(\hat{u})(\partial_t \hat{u} + X_{\hat{\Psi}}(\hat{u})) = 0 \\
\partial_s \hat{\Psi} - \partial_t \hat{\Phi} + [\hat{\Phi}, \hat{\Psi}] + \mu(\hat{u}) = 0 \\
(\hat{u}, \hat{\Psi}, \hat{\Phi})(s+2, t) = [(R_0) \circ (R_1)\hat{u}, (\dot{S}_0) \circ (\dot{S}_1)\hat{\Psi}, (\dot{S}_0) \circ (\dot{S}_1)\hat{\Phi})(s, t).
\end{gathered} \tag{16}$$

Solutions of (16) are invariant under the action of the gauge group

$$\hat{\mathcal{G}}_{loc} := \{g \in C^{\infty}_{loc}(\mathbb{C}, G) : g(s, t+2) = (S_0) \circ (S_1) g(s,t)\}.$$

4.3 Compactness

The energy of a solution of (14) is defined by

$$E(u, \Psi, \Phi) := \int_{\Theta} \left(|\partial_s u + X_{\Phi}(u)|^2 + |\mu(u)|^2 \right) ds dt.$$

The aim of this subsection is to prove that every sequence of finite energy solutions of (14) has a convergent subsequence modulo gauge invariance. The main ingredient in the proof is Uhlenbeck's compactness theorem, which states that a connection with an L^p-bound on the curvature is gauge equivalent to a connection which satisfies an L^p-bound on all its first derivatives.

Compactness fails if solutions of (14) can escape to infinity. To make sure that this cannot happen we have to choose our almost complex structure and the Hamiltonian function appropriately. Fix some convex structure $\mathcal{K} = (f, \tilde{J})$ on $(M, \omega, \mu, L_1, L_2)$. Let $\mathcal{J}(M, \omega, \mu, \mathcal{K})$ be the space of all G-invariant ω-compatible almost complex structures J on (M, ω) which equal $\tilde{J}$ outside of a compact set in M. It is proven in Proposition 2.50 in [MS1] that the space $\mathcal{J}(M, \omega, \mu, \mathcal{K})$ is nonempty and contractible. We define the **space of admissible families of almost complex structures**

$$\mathcal{J} := \mathcal{J}([0,1], M, \omega, \mu, \mathcal{K}) \subset C^{\infty}([0,1], \mathcal{J}(M, \omega, \mu, \mathcal{K}))$$

as the space consisting of smooth families of $J_t \in \mathcal{J}(M, \omega, \mu, \mathcal{K})$ which satisfy (15). Let $C^{\infty}_{0,G}(M)$ be the space of smooth G-invariant functions on M with compact support, and

$$\mathrm{Ham} := \mathrm{Ham}(M, G) := \{H \in C^{\infty}_0([0,1] \times M) : H_t \in C^{\infty}_{0,G}(M)\}$$

the space of G-invariant functions parametrised by $t \in [0,1]$.

Theorem 4.13 (Compactness) *Let* $(u_\nu, \Psi_\nu, \Phi_\nu) \in C^\infty_{loc}(\Theta, M \times \mathfrak{g} \times \mathfrak{g})$ *be a sequence of solutions of (14) with respect to a smooth family of almost complex structures* $J_t \in \mathcal{J}$ *and to a smooth family of Hamiltonian functions* $H_t \in \mathrm{Ham}$. *If the energies are uniformly bounded, then there exists a sequence of gauge transformations* $g_\nu \in \mathcal{G}_{loc}$ *such that a subsequence of* $(g_\nu)_*(u_\nu, \Psi_\nu, \Phi_\nu)$ *converges in the* C^∞_{loc}*-topology to a smooth solution* (u, Ψ, Φ) *of the vortex problem(14).*

Instead of Theorem 4.13 we prove the following stronger theorem.

Theorem 4.14 *Let* $(u_\nu, \Psi_\nu) \in C^\infty_{loc}(\Theta, M \times \mathfrak{g})$ *be a sequence of solutions of (13) with respect to a smooth family of almost complex structures* $J_t \in \mathcal{J}$ *and to a smooth family of Hamiltonian functions* $H_t \in \mathrm{Ham}$. *If the energies are uniformly bounded, then there exists a sequence of gauge transformations* $g_\nu \in \mathcal{H}$ *such that a subsequence of* $(g_\nu)_*(u_\nu, \Psi_\nu)$ *converges in the* C^∞_{loc}*-topology to a smooth solution* (u, Ψ) *of the gradient equation (13).*

Proof: Let $(\hat{u}_\nu, \hat{\Psi}_\nu) \in C^\infty_{loc}(\mathbb{C}, M \times \mathfrak{g})$ be the extension of (u_ν, Ψ_ν) as in Remark 4.12. We will prove the theorem in four steps.

Step 1: *For every compact subset* K *of* $\mathbb{C}$ *there exists a sequence of gauge transformations* $g_\nu \in C^\infty(K, G)$ *such that* $(g_\nu)_*(\hat{u}_\nu, \hat{\Psi}_\nu, 0)|_K$ *converges in the* C^∞*-topology.*

Step 1 was proved in [CGMS, Theorem 3.4.]. The main ideas are the following. Let $\hat{A}_\nu = \hat{\Psi}_\nu dt$ be the connection on the trivial G-bundle over $\mathbb{C}$. By convexity $\hat{u}_\nu(K)$ is contained in a compact subset of M. The curvature of $\hat{A}_\nu$ is given by $F_{\hat{A}_\nu} = \partial_s \hat{\Psi}_\nu ds \wedge dt$. Hence the equation $\partial_s \hat{\Psi}_\nu + \mu(\hat{u}_\nu) = 0$ implies that the curvature is uniformly bounded. Moreover, because of (H4) there is no bubbling and hence

$$\sup_\nu ||\partial_s \hat{u}_\nu|_K||_\infty < \infty.$$

Step 1 follows now from a combination of Uhlenbecks compactness theorem, see [Uh, We], and the Compactness Theorem for the Cauchy-Riemann operator (see for example [MS2, Appendix B]).

Step 2: *There exists a sequence of gauge transformations* $g_\nu \in C^\infty_{loc}(\mathbb{C}, G)$ *such that a subsequence of* $(g_\nu)_*(\hat{u}_\nu, \hat{\Psi}_\nu, 0)$ *converges in the* C^∞_{loc}*-topology.*

We use the fact that $\mathbb{C}$ can be exhausted by compact sets, $\mathbb{C} = \bigcup_{n \in \mathbb{N}} B_n$ where $B_n := \{z \in \mathbb{C} : |z| = n\}$. Let g^n_ν be the sequence of gauge transformations on B_n for $n \in \mathbb{N}$ obtained in step 1. We show that we may assume that

$$g^n_\nu|_{B_{n-1}} = g^{n+1}_\nu|_{B_{n-1}}, \quad \forall\ \nu \in \mathbb{N}. \tag{17}$$

To prove (17) we first observe that there exists $h^n \in C^\infty(B_n, G)$ such that $h^n_\nu := (g^{n+1}_\nu)^{-1} \circ g^n_\nu|_{B_n}$ has a subsequence which converges in the C^∞-topology

to h^n. Hence there exists a sequence $\tilde{h}^n_\nu$ which has a converging subsequence and satisfies

$$\tilde{h}^n_\nu(z) := \begin{cases} h^n_\nu(z) & z \in B_{n-1} \\ \mathrm{id} & z \in B_{n+1} \setminus B_n. \end{cases}$$

Now replace g^{n+1}_ν by $g^{n+1}_\nu \circ \tilde{h}^n_\nu$, which satisfies (17). Now define g_ν by

$$g_\nu|_{B_n} := g^{n+1}_\nu|_{B_n}, \quad \forall\, n \in \mathbb{N}.$$

Step 3: *The sequence of gauge transformations $g_\nu(s+it) \in C^\infty_{loc}(\mathbb{C}, G)$ in step 2 may be chosen independent of s.*

We use an idea from [JRS]. Denote the limit of the sequence $(g_\nu)_*(\hat{u}_\nu, \hat{\Psi}_\nu, 0)$ as ν goes to infinity by $(\hat{u}, \hat{\Psi}, \hat{\Phi})$. Choose $h \in C^\infty_{loc}(\mathbb{C}, G)$ such that

$$h_* \hat{\Phi} = 0.$$

Observe that

$$\lim_{\nu \to \infty} (\partial_s (h \circ g_\nu))(h \circ g_\nu)^{-1}) = \lim_{\nu \to \infty} (h \circ g_\nu)_*(0) = h_* \hat{\Phi} = 0.$$

It follows that $\partial_s (h \circ g_\nu)$ converges to zero in the C^∞_{loc}-topology. Now set

$$\tilde{g}_\nu(s,t) := h \circ g_\nu(0,t).$$

Obviously, $\tilde{g}_\nu$ is independent of s. Moreover, since $\tilde{g}_\nu \circ (h \circ g_\nu)^{-1}$ converges to the identity in the C^∞_{loc}-topology, $\tilde{g}_\nu$ satisfies the assumptions of step 2. We denote $\tilde{g}_\nu$ by g_ν as before.

Step 4: *We prove the theorem.*

We have to modify further our sequence of gauge transformations g_ν, such that they satisfy the boundary conditions. Because the energy of the sequence (u_ν, Ψ_ν) is bounded, the energy of $(\hat{u}, \hat{\Psi})|_\Theta$ is bounded. Using the fact that finite energy solutions converge uniformly at the two ends of the strip, see [Fr2], we conclude that $\mu(\hat{u})(s,t)$ converges to zero as s goes to infinity uniformly in the t-variable. Since G acts freely on $\mu^{-1}(0)$, there exists $s_0 \in \mathbb{R}$ such that

$$G_{u(s_0,0)} = G_{u(s_0,1)} = \{\mathrm{id}\}.$$

Since by assumption $\hat{u}_\nu(s_0, j) \in L_j$ for $j \in \{0,1\}$ it follows that $\hat{u}(s_0,j) \in GL_j$. Now choose $h \in C^\infty([0,1], G)$ such that $h(0)\hat{u}(s_0,0) \in L_0$ and $h(1)\hat{u}(s_0,1) \in L_1$. This is possible, because G is connected. Choose a sequence $h_\nu \in C^\infty([0,1], G)$ converging to the identity satisfying

$$(h_\nu(j) \circ h(j) \circ g_\nu(j)) u_\nu(s_0, j) \in L_j, \quad j \in \{0,1\}.$$

We can assume without loss of generality that

$$G_{u_\nu(s_0,0)} = G_{u_\nu(s_0,1)} = \{\mathrm{id}\}$$

for every ν. By Lemma 4.4, $h_\nu(j) \circ h(j) \circ g_\nu(j) \in G_{L_j}$ for $j \in \{0,1\}$. In particular,

$$(h_\nu \circ h \circ g_\nu) u_\nu(s,j) \in L_j, \quad j \in \{0,1\}.$$

Hence $h_* \hat{u}(s,j) \in L_j$ for $j \in \{0,1\}$. By a similar procedure as above we may find a further sequence of gauge transformations $\tilde{h}_\nu \in C^\infty([0,1],G)$ which satisfy the following conditions.

(i) $\tilde{h}_\nu(j) \in G_{L_j}$ for $j \in \{0,1\}$,

(ii) $(\tilde{h}_\nu)_*(g_\nu)_* h_* \hat{\Psi}_\nu(s,j) \in \mathfrak{g}^\perp_{L_j}$ for every ν and $j \in \{0,1\}$,

(iii) There exists $\tilde{h} \in C^\infty([0,1],G)$ such that $\tilde{h}_\nu$ converges as ν goes to infinity in the C^∞-topology to $\tilde{h}$.

Now set

$$\tilde{g}_\nu := \tilde{h}_\nu \circ h_\nu \circ h \circ g_\nu|_{[0,1]} \in C^\infty([0,1],G).$$

Moreover, using the assumption that $\hat{\Psi}_\nu(j) \in \mathfrak{g}^\perp_{L_j}$ for $j \in \{0,1\}$, one calculates

$$\tilde{g}_\nu(j) \in G_{L_j}, \quad \tilde{g}_\nu^{-1} \partial_t \tilde{g}_\nu(j) \in \mathfrak{g}^\perp_{L_j}, \quad j \in \{0,1\},$$

and hence $\tilde{g}_\nu \in \mathcal{H}$. Set

$$(u,\Psi) := \tilde{h}_* h_* (\hat{u},\hat{\Psi})|_\Theta.$$

Now the theorem follows with g_ν replaced by $\tilde{g}_\nu$. □

4.4 Moduli spaces

We assume that the Hamiltonian $H \in \mathrm{Ham}$ has the property that the Lagrangians $\phi^1_{\bar{H}}(\bar{L}_0)$ and $\bar{L}_1$ in the Marsden-Weinstein quotient intersect transversally. Under this assumption it can be shown, see [Fr2], that finite energy solutions of the symplectic vortex equation (14) are gauge equivalent to solutions which decay exponentially fast at the two ends of the strip. More precisely, define for some small number $\delta > 0$ and some smooth cutoff function β satisfying $\beta(s) = -1$ if $s < 0$ and $\beta(s) = 1$ if $s > 1$

$$\gamma_\delta \in C^\infty(\mathbb{R}), \quad s \mapsto e^{\delta\beta(s)s}.$$

For an open subset $S \subset \Theta$ we define the $|| \, ||_{C^k_\delta}$-norm for some smooth function $f\colon S \to \mathbb{R}$ by

$$||f||_{C^k_\delta} := ||\gamma_\delta \cdot f||_{C^k}$$

and denote

$$C^\infty_\delta(S) := \{f \in C^\infty(S)\colon ||f||_{C^k_\delta} < \infty, \ \forall\, k \in \mathbb{N}\}. \tag{18}$$

We now introduce the Fréchet manifold $\mathcal{B} = \mathcal{B}_\delta$ as the set consisting of all $w = (u,\Psi,\Phi) \in C^\infty_{loc}(\Theta, M \times \mathfrak{g}) \times C^\infty_\delta(\Theta,\mathfrak{g})$ which satisfy the following conditions:

(i) w maps (s,j) to $L_j \times \mathfrak{g}_{L_j} \times \mathfrak{g}_{L_j}^\perp$ for $j \in \{0,1\}$ and $s \in \mathbb{R}$.

(ii) There exists a critical point of the action functional $(x_1, \eta_1) \in \mathrm{crit}(\mathcal{A})$, a real number $T_1 \in \mathbb{R}$, and $(\xi_1, \psi_1) \in C^\infty_\delta((-\infty, T_1] \times [0,1], x_1^*TM \times \mathfrak{g})$ such that

$$(u, \Psi)(s,t) = (\exp_{x_1(t)}(\xi_1(s,t)), \eta_1(t) + \psi_1(s,t)), \quad s \leq -T_1.$$

(iii) There exists a critical point of the action functional $(x_2, \eta_2) \in \mathrm{crit}(\mathcal{A})$, a real number $T_2 \in \mathbb{R}$, and $(\xi_2, \psi_2) \in C^\infty_\delta([T_2, \infty) \times [0,1], x_2^*TM \times \mathfrak{g})$ such that

$$(u, \Psi)(s,t) = (\exp_{x_2(t)}(\xi_2(s,t)), \eta_2(t) + \psi_2(s,t)).$$

The theorem about exponential decay proved in [Fr2] now tells us, that for $\delta > 0$ chosen small enough every finite energy solution of the symplectic vortex equations is gauge equivalent to an element in $\mathcal{B}_\delta$. Moreover, one can prove that every solution of (14) which lies in $\mathcal{B}_\delta$ has finite energy.

In a similar vein we define the gauge group $\mathcal{G} = \mathcal{G}_\delta$ consisting of gauge transformations $g \in \mathcal{G}_{loc}$ which decay exponentially fast at the two ends of the strip to elements of $\mathcal{H}_0$ the connected component of the identity of the gauge group $\mathcal{H}$. Note that there are natural evaluation maps

$$\mathrm{ev}_1, \mathrm{ev}_2 \colon \frac{\{(14)\} \cap \mathcal{B}}{\mathcal{G}} \to \pi_0(\mathrm{crit}(\mathcal{A})) \cong (\phi^1_{\bar{H}}(\bar{L}_0) \cap \bar{L}_1) \times \pi_0(\mathcal{H}),$$

induced by the maps $w \mapsto (x_1, \eta_1)$ and $w \mapsto (x_2, \eta_2)$. One can check that the energy of an element $w \in (\{(14)\} \cap \mathcal{B})/\mathcal{G}$ is given by the difference of the actions

$$E(w) = \mathcal{A}(\mathrm{ev}_1(w)) - \mathcal{A}(\mathrm{ev}_2(w))^6.$$

It can be shown, see [Fr2] that the linearization of the symplectic vortex equations considered as an operator between suitable Banach spaces is a Fredholm operator. Moreover, using hypothesis (H3) it follows that the path space $\mathscr{P}$ defined in (11) is connected and simply connected. This implies that there exists a function

$$I \colon \pi_0(\mathrm{crit}(\mathcal{A})) \to \mathbb{Z}$$

such that the Fredholm index of the linearized symplectic vortex equations at a point $w \in \mathcal{B}/\mathcal{G}$ is given by the difference $I(\mathrm{ev}_1(w)) - I(\mathrm{ev}_2(w))$.

We can now introduce on $\pi_0(\mathrm{crit}(\mathcal{A}))$ the following equivalence relation. We say that two connected components of $\mathrm{crit}(\mathcal{A})$ are equivalent if they project to the same intersection point of $\phi^1_{\bar{H}}(\bar{L}_0) \cap \bar{L}_1$ and the action functional $\mathcal{A}$ and the index I agree on them. We denote the set of such equivalence classes by $\mathscr{C} = \mathscr{C}(\mathcal{A})$. For $c_1, c_2 \in \mathscr{C}$ we define the moduli space

$$\tilde{\mathcal{M}}(c_1, c_2) := \{w \in (\{14\} \cap \mathcal{B})/\mathcal{G} : \mathrm{ev}_1(w) \in c_1,\ \mathrm{ev}_2(w) \in c_2\}. \qquad (19)$$

[6] Since the action functional is invariant under the action of $\mathcal{H}_0$ we denote by abuse of notation the function induced from the action functional on $\pi_0(\mathrm{crit}(\mathcal{A}))$ also by $\mathcal{A}$.

Note that the moduli spaces depend on the choice of the almost complex structure $J \in \mathcal{J}$, i.e. $\tilde{\mathcal{M}}(c_1, c_2) = \tilde{\mathcal{M}}_J(c_1, c_2)$. It is proved in [Fr2] that for generic choice of the almost complex structure the Fredholm operators obtained by linearizing the symplectic vortex equations are surjectiv. In particular, the moduli spaces $\tilde{\mathcal{M}}_J(c_1, c_2)$ are smooth manifolds whose dimension is given by the difference of the Fredholm indices of c_1 and c_2.

Theorem 4.15 *There exists a subset $\mathcal{J}_{reg} \subset \mathcal{J}$ of second category such that $\tilde{\mathcal{M}}_J(c_1, c_2)$ for every $c_1, c_2 \in \mathscr{C}$ are smooth finite dimensional manifolds whose dimension is given by*

$$\dim(\tilde{\mathcal{M}}_J(c_1, c_2)) = I(c_1) - I(c_2).$$

The group $\mathbb{R}$ acts on $\tilde{\mathcal{M}}(c_1, c_2)$ by timeshift We define the path space $\mathscr{P}$ by

$$w(s,t) \mapsto w(s+r, t), \quad r \in \mathbb{R}.$$

If $c_1 \neq c_2$ then this action is free and the quotient $\tilde{\mathcal{M}}(c_1, c_2)/\mathbb{R}$ is again a manifold. Using the compactness result in Theorem 4.13 one can show as in [S2] that the only obstruction to compactness of the spaces $\tilde{\mathcal{M}}(c_1, c_2)$ is breaking off of flow lines, see [Fr2].

4.5 A Novikov ring

If $c_1, c_2 \in \pi_0(\mathrm{crit}(\mathcal{A}))$ and $h \in \mathcal{H}$ then

$$\mathcal{A}(c_1) - \mathcal{A}(c_2) = \mathcal{A}(hc_1) - \mathcal{A}(hc_2), \quad I(c_1) - I(c_2) = I(hc_1) - I(hc_2)$$

where I is the Fredholm-index introduced in the previous subsection. It follows that $I(hc) - I(c)$ and $\mathcal{A}(hc) - \mathcal{A}(c)$ is independent of the choice of $c \in \pi_0(\mathrm{crit}(\mathcal{A}))$. Hence we may define the maps

$$I_{\mathcal{H}} \colon \mathcal{H} \to \mathbb{Z}, \quad E_{\mathcal{H}} \colon \mathcal{H} \to \mathbb{R}, \quad h \mapsto I(hc) - I(c), \quad h \mapsto \mathcal{A}(hc) - \mathcal{A}(c)$$

for some arbitrary $c \in \pi_0(\mathrm{crit}(\mathcal{A}))$. One easily checks that these maps are group homomorphisms. Moreover they vanish on $\mathcal{H}_0$, the connected component of the identity of $\mathcal{H}$. For the special case where $M = \mathbb{C}^n$, $L_0 = L_1 = \mathbb{R}^n$, and G acts on $\mathbb{C}^n$ by a linear injective representation ρ one can show, see [Fr2], that the index map $I_{\mathcal{H}}$ is given by

$$I_{\mathcal{H}}(h) = \deg(\det{}^2_{\mathbb{C}}(\rho(h)))$$

for $h \in \mathcal{H}$.

We define

$$\Gamma = \frac{\mathcal{H}}{\ker I_{\mathcal{H}} \cap \ker E_{\mathcal{H}}}.$$

To the group Γ we associate the Novikov ring $\Lambda = \Lambda_\Gamma$ whose elements are formal sums

$$r = \sum_{\gamma \in \Gamma} r_\gamma \gamma$$

with coefficients $r_\gamma \in \mathbb{Z}_2$ which satisfy the finiteness condition

$$\#\{\gamma \in \Gamma : r_\gamma \neq 0, E_{\mathcal{H}}(\gamma) \geq \kappa\} < \infty$$

for every $\kappa > 0$. The multiplication is given by

$$r * s = \sum_{\gamma \in \Gamma} \Bigg(\sum_{\substack{\gamma_1, \gamma_2 \in \Gamma \\ \gamma_1 \circ \gamma_2 = \gamma}} r_{\gamma_1} s_{\gamma_2} \Bigg) \gamma.$$

Since the coefficients r_γ are taken in a field, the Novikov ring is actually a field. The ring comes with a natural grading defined by

$$\deg(\gamma) = I_{\mathcal{H}}(\gamma)$$

and we shall denote by Λ_k the elements of degree k. Note in particular that Λ_0 is a subfield of Λ. Moreover, the multiplication maps $\Lambda_j \times \Lambda_k \to \Lambda_{j+k}$.

4.6 Definition of the homology

We assume that $H \in \mathrm{Ham}$ has the property that $\phi^1_{\bar{H}}(\bar{L}_0)$ and $\bar{L}_1$ intersect transversally and $J \in \mathcal{J}_{reg} = \mathcal{J}_{reg}(H)$, i.e. the Fredholm operators obtained by linearizing the symplectic vortex equations are surjective. Recall

$$\mathscr{C} = \mathscr{C}(\mathcal{A}) = \frac{\mathrm{crit}(\mathcal{A})}{\ker I_{\mathcal{H}} \cap \ker E_{\mathcal{H}}} \cong (\phi^1_{\bar{H}}(\bar{L}_0) \cap \bar{L}_1) \times \Gamma. \tag{20}$$

We define the chain complex $CF_*(H, L_0, L_1, \mu)$ as a module over the Novikov ring Λ. More precisely, $CF_k(H, L_0, L_1, \mu)$ are formal sums of the form

$$\xi = \sum_{\substack{c \in \mathscr{C} \\ I(c) = k}} \xi_c c$$

with $\mathbb{Z}_2$-coefficients ξ_c satisfying the finiteness condition

$$\#\{c : \xi_c \neq 0, E(c) \geq \kappa\} < \infty \tag{21}$$

for every $\kappa > 0$. The action of Γ on $\mathscr{C}$ is the induced action of $\mathcal{H}$ on $\mathrm{crit}(\mathcal{A})$. The Novikov ring acts on CF_* by

$$r * \xi = \sum_{c \in \mathscr{C}} \sum_{\substack{c' \in \mathscr{C}, \gamma' \in \Gamma \\ \gamma' c' = c}} \Big(r_{\gamma'} \xi_{\gamma'} \Big) c.$$

CF_k is invariant under the action of Λ_0. In particular, CF_k which may be an infinite dimensional vector space over the field $\mathbb{Z}_2$, is a finite dimensional vector space over the field Λ_0.

Recall that for $c_1, c_2 \in \mathscr{C}$ the moduli space is defined by

$$\tilde{\mathcal{M}}(c_1, c_2) := \{w \in (\{14\} \cap \mathcal{B})/\mathcal{G} : \mathrm{ev}_1(w) \in c_1,\ \mathrm{ev}_2(w) \in c_2\}.$$

Let $\mathcal{T}$ be the group

$$\mathcal{T} := \frac{\ker I_{\mathcal{H}} \cap \ker E_{\mathcal{H}}}{\mathcal{H}_0},$$

where $\mathcal{H}_0$ is the connected component of the identity of $\mathcal{H}$. Then $\mathcal{T} \times \mathbb{R}$ act on $\tilde{\mathcal{M}}$ by

$$w(s) \mapsto g_* w(s+r), \quad (g,r) \in \mathcal{T} \times \mathbb{R}$$

and we define

$$\mathcal{M}(c_1, c_2) := \frac{\tilde{\mathcal{M}}(c_1, c_2)}{\mathcal{T} \times \mathbb{R}}.$$

Assume that $c_1 \neq c_2$. Under this assumption $\mathcal{T} \times \mathbb{R}$ acts freely on $\tilde{\mathcal{M}}(c_1, c_2)$ and since $J \in \mathcal{J}_{reg}$ the moduli spaces are manifolds of dimension

$$\dim \mathcal{M}(c_1, c_2) = \dim \tilde{\mathcal{M}}(c_1, c_2) - 1 = I(c_1) - I(c_2) - 1.$$

Using the fact that the only obstruction to compactness for strips of finite energy is the breaking off phenomenon, which cannot happen in the case where the index equals zero, we conclude that for $c_1, c_2 \in \mathscr{C}$ with $I(c_1) - I(c_2) = 1$ and $\kappa > 0$ we have

$$\sum_{\substack{\gamma \in \Gamma \\ E_{\mathcal{H}}(\gamma) \geq \kappa,\ I_{\mathcal{H}}(\gamma) = 0}} \#\mathcal{M}(c_1, \gamma c_2) < \infty. \tag{22}$$

Set

$$n(c_1, c_2) := \#\mathcal{M}(c_1, c_2) \mod 2$$

and define the boundary operator $\partial_k : CF_k \to CF_{k-1}$ as linear extension of

$$\partial_k c = \sum_{I(c') = k-1} n(c, c') c'$$

for $c \in \mathscr{C}$ with $I(c) = k$. Note that (22) guarantees the finiteness condition (21) for $\partial_k c$.

As in the standard theory (see [Sch1, Sch2, HS] one shows that

$$\partial^2 = 0.$$

This gives rise to homology groups

$$HF_k(H, J, L_0, L_1, \mu; \Lambda) := \frac{\ker \partial_{k+1}}{\mathrm{im} \partial_k}.$$

A standard argument (see [Sch1] or [HS]) shows that $HF_k(H, J, , L_0, L_1, \mu; \Lambda)$ is actually independent of the regular pair (H, J). Hence we set for some regular pair (H, J)

$$HF_k(L_0, L_1, \mu; \Lambda) := HF_k(H, J, L_0, L_1, \mu; \Lambda).$$

We call the graded Λ vector space $HF_*(L_0, L_1, \mu; \Lambda)$ the **moment Floer homology**.

4.7 Computation of the homology

In this subsection we compute moment Floer homology for case where the two Lagrangians coincide, i.e. $L_0 = L_1 = L$. We will see that in this case, the moment Floer homology equals the singular homology of the induced Lagrangian in the Marsden-Weinstein quotient $\bar{L}$ tensored with the Novikov ring introduced above. As a corollary we get a proof of the Arnold-Givental conjecture for $\bar{L}$.

Theorem 4.16 *Assume that the two Lagrangians coincide, i.e. $L_0 = L_1 = L$, then*

$$HF_*(L, \mu; \Lambda) := HF_*(L, L, \mu; \Lambda) = H\bar{L}_*(\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda.$$

Corollary 4.17 *The Arnold-Givental conjecture holds for $\bar{L}$, i.e. under the transversality assumption $\bar{L} \pitchfork \phi^1_{\bar{H}} \bar{L}$,*

$$\#(\bar{L} \cap \phi^1_{\bar{H}} \bar{L}) \geq \sum_k b_k(\bar{L}, \mathbb{Z}_2).$$

To prove Theorem 4.16 we consider the case where the Hamiltonian H vanishes. In this case the Lagrangians in the quotient $\bar{L}$ and $\phi^1_{\bar{H}}(\bar{L}) = \bar{L}$ coincide. In particular, they do not intersect transversally but still cleanly, i.e. there intersection is still a manifold whose tangent space is given by the intersection of the two tangent spaces. This is the infinite dimensional analogon of a Morse-Bott situation. In our case the critical manifold can be identified with $\bar{L} \times \Gamma$. Following the approach explained in the appendix, we still can define the homology in this case by choosing a Morse function on the critical manifold. To define the boundary operator one has to count flow lines with cascades. There is a natural splitting of the boundary operator into two parts. The first part takes account of the flow lines with zero cascades, i.e. Morse flow lines on the critical manifold, and the second part takes account of flow lines with at least one cascade. To prove the theorem we have to show that the second part of the boundary operator vanishes. Using the antisymplectic involution we construct successive involutions on the cascades which endowes the space of cascades with the structure of a space of Arnold-Givental type which admits the structure of a Kuranishi structure of stable Arnold-Givental type. Using this it follows that the second part of the boundary vanishes.

We now define moment Floer homology for the case where the Hamiltonian $H = 0$. We think of $\mathcal{H}(\mu^{-1}(0) \cap L)$ as the critical manifold of the action functional $\mathcal{A} = \mathcal{A}_0$ of the unperturbed symplectic vortex equations. A Morse function on the induced Lagrangian in the Marsden-Weinstein quotient $\bar{L} = \mu^{-1}(0)/G_L$ will lift to a $\mathcal{H}$-invariant Morse function on the critical manifold of $\mathcal{A}$.

We first describe the elements which are needed to define the chain complex. Choose a Riemannian metric g and a Morse-function f on $\bar{L}$ which satisfy the Morse-Smale condition, i.e. stable and unstable manifolds intersect transversally, and lift it to a G_L-equivariant metric $\tilde{g}$ and a G_L-equivariant Morse-function $\tilde{f}$ on $\mu^{-1}(0) \cap L$. Recall that for $x \in M$ and $\eta \in \mathfrak{g}$ the linear map

$L_x\colon \mathfrak{g} \to T_xM$ was defined by

$$L_x\eta = X_\eta(x) = \frac{d}{dr}\bigg|_{r=0} \exp(r\eta)(x).$$

Let $\tilde{\mathscr{C}_0} = \tilde{\mathscr{C}_0}(f)$ be the set of smooth maps $(x, \eta) : [0,1] \to M \times \mathfrak{g}$ satisfying

$$\dot{x}(t) + L_{x(t)}\eta(t) = 0,\ \mu(x(t)) = 0,\ t \in [0,1],$$

$$x(j) \in \mathrm{crit}(\tilde{f}),\ \eta(j) \in \mathfrak{g}_L^\perp,\ j \in \{0,1\}.$$

Note that η is completely determined by x through the formula

$$\eta(t) = -(L^*_{x(t)}L_{x(t)})^{-1}L^*_{x(t)}\dot{x}(t),$$

where L^*_x is the adjoint of L_x with respect to the fixed invariant inner product on $\mathfrak{g}$ and the inner product $\omega_x(\cdot, J_t(x)\cdot)$ on $T_{x(t)}M$. Moreover, it follows from Proposition 4.11 that there exists an element g_x of the gauge group $\mathcal{H}$ such that

$$x(t) = g_x(t)x(0).$$

Denote

$$\mathscr{C}_0 := \mathscr{C}_0(f) := \frac{\tilde{\mathscr{C}_0}}{\ker I_{\mathcal{H}} \cap \ker E_{\mathcal{H}}}. \tag{23}$$

Then the map

$$(x, \eta) \mapsto (x(0), g_x)$$

defines a natural bijection

$$\tilde{\mathscr{C}_0} \cong \mathrm{crit}(\tilde{f}) \times \mathcal{H}$$

and induces a bijection in the quotient

$$\mathscr{C}_0 \cong \frac{\mathrm{crit}(\tilde{f})}{G_L} \times \Gamma \cong \mathrm{crit}(f) \times \Gamma.$$

If ind_f is the Morse-index, then the index of a critical point $c = (q,h) \in \tilde{\mathscr{C}_0}$ is defined to be

$$I(q,h) := \mathrm{ind}_f(\pi(q)) + I_{\mathcal{H}}(h)$$

for the canonical projection onto the Marsden-Weinstein quotient $\pi : \mu^{-1}(0) \to \bar{M} = \mu^{-1}(0)/G$. We define the energy of a critical point by

$$E(q,h) := E_{\mathcal{H}}(h).$$

By abuse of notation we will also denote by I and E the induced index and the induced energy on the quotient $\mathscr{C}_0$.

We next introduce flow lines with cascades which are needed to define the boundary operator.

Definition 4.18 *For* $c_1 = (q_1, h_1), c_2 = (q_2, h_2) \in \tilde{\mathscr{C}_0}$ *and* $m \in \mathbb{N}$ *a* **flow line from** c_1 **to** c_2 **with** m **cascades**

$$v = ((w_k)_{1 \leq k \leq m}, (T_k)_{1 \leq k \leq m-1}) =$$

$$((u_k, \Psi_k, \Phi_k)_{1 \leq k \leq m}, (T_k)_{1 \leq k \leq m-1})$$

consists of the triple of functions $(u_k, \Psi_k, \Phi_k) \in C^\infty_{loc}(\Theta, M \times \mathfrak{g}) \times C^\infty_\delta(\Theta, \mathfrak{g})$ [7] *and the nonnegative real numbers* $T_k \in \mathbb{R}_\geq := \{r \in \mathbb{R} : r \geq 0\}$ *which have the following properties:*

(i) (u_k, Ψ_k, Φ_k) *are nonconstant, finite energy solutions of (14) with Hamiltonian equal to zero, i.e.*

$$\begin{aligned} \partial_s u_k + X_{\Phi_k}(u_k) + J_t(u_k)(\partial_t u_k + X_{\Psi_k}(u_k)) = 0 \\ \partial_s \Psi_k - \partial_t \Phi_k + [\Phi_k, \Psi_k] + \mu(u_k) = 0 \\ u_k(s,j) \in L, \quad \Phi_k(s,j) \in \mathfrak{g}_L, \quad \Psi_k(s,j) \in \mathfrak{g}_L^\perp, \end{aligned} \tag{24}$$

where $j \in \{0, 1\}$.

(ii) *There exist points* $p_1 \in W^u_{\tilde{f}}(q_1)$ *and* $p_2 \in W^s_{\tilde{f}}(q_2)$ *such that* $\lim_{s \to -\infty} u_1(s,t) = h_1(t)p_1$ *and* $\lim_{s \to \infty} u_m(s,t) = h_2(t)p_2$ *uniformly in the* t*-variable.*

(iii) *For* $1 \leq k \leq m-1$ *there exist Morse-flow lines* $y_k : (-\infty, \infty) \to \mu^{-1}(0) \cap L$, *i.e. solutions of*

$$\dot{y}_k = -\nabla_{\tilde{g}} \tilde{f}(y_k),$$

and $g_k \in \mathcal{H}$ *such that*

$$\lim_{s \to \infty} u_k(s,t) = g_k(t) y_k(0)$$

and

$$\lim_{s \to -\infty} u_{k+1}(s,t) = g_k(t) y_k(T_k),$$

where the two limites are uniform in the t*-variable.*

A **flow line with zero cascades from** $c_1 = (q_1, h)$ **to** $c_2 = (q_2, h)$ *is a tuple* (y, h) *where* y *is just an ordinary Morse flow line from* q_1 *to* q_2.

Rccall from p. 46 that the gauge group $\mathcal{G}$ consists of smooth maps from the strip to G, which satisfy appropriate boundary conditions and which decay exponentially. For $m \in \mathbb{N}$ the group $\mathcal{G}_m$ consists of m-tuples

$$\mathbf{g} = (g_k)_{1 \leq k \leq m}$$

where $g_k \in \mathcal{G}$, which have the additional property that they form a chain, i.e.

$$\mathrm{ev}_2(g_k) = \mathrm{ev}_1(g_{k+1}), \quad 1 \leq k \leq m-1.$$

[7] See (18) for the definition of the space C^∞_δ.

For $m \geq 1$ the group $\mathcal{G}_m \times \mathbb{R}^m$ acts on the space of flow lines with m cascades as follows

$$(u_k, \Psi_k, \Phi_k)(s) \mapsto (g_k)_*(u_k, \Psi_k, \Phi_k)(s + s_k)$$

for $1 \leq k \leq m$ and $(g_k, s_k) \in \mathcal{G} \times \mathbb{R}$. For $m = 0$ the group $\frac{\mathcal{H}}{\ker I_{\mathcal{H}} \cap \ker E_{\mathcal{H}}} \times \mathbb{R}$ acts on the space of flow lines with zero cascades by

$$(y(s), h) \mapsto (y(s + s_0), g \circ h).$$

For $c_1, c_2 \in \mathscr{C}_0$ we denote the quotient of flow lines with m cascades from c_1 to c_2 for $m \in \mathbb{N}_0$ by

$$\mathcal{M}_m(c_1, c_2).$$

We define the **space of flow lines with cascades from** c_1 **to** c_2 by

$$\mathcal{M}(c_1, c_2) := \bigcup_{m \in \mathbb{N}_0} \mathcal{M}_m(c_1, c_2).$$

Using the transversality result for the symplectic vortex equations in [Fr2], one proves in the same way as Theorem A.11 that the moduli spaces of flow lines with cascades are finite dimensional manifolds for generic choice of the almost complex structure.

Theorem 4.19 *For each pair of a Morse function f on $\bar{L} = (\mu^{-1}(0) \cap L)/G_L$ and a Riemannian metric g on $\bar{L}$ which satisfy the Morse-Smale condition, i.e. its stable and unstable manifolds intersect transversally, there exists a subset of the space of admissible families of almost complex structures*

$$\mathcal{J}_{reg} = \mathcal{J}_{reg}(f, g) \subset \mathcal{J}$$

which is of the second category, i.e. $\mathcal{J}_{reg}$ is a countable intersection of open and dense subsets of $\mathcal{J}$, and which is regular in the following sense. For any two critical points $c_1, c_2 \in \mathscr{C}_0$ the space $\mathcal{M}(c_1, c_2) = \mathcal{M}(c_1, c_2; J, f, g)$ is a smooth finite dimensional manifold. Its dimension is given by

$$\dim(\mathcal{M}(c_1, c_2)) = I(c_1) - I(c_2) - 1.$$

If $I(c_1) - I(c_2) - 1 = 0$, then $\mathcal{M}(c_1, c_2)$ is compact and hence a finite set.

We are now able to define moment Floer homology in the case where the Hamiltonian vanishes. Choose a triple (f, g, J) where f is a Morse function on $\bar{L} = (\mu^{-1}(0) \cap L)/G_L$, g is a Riemannian metric on $\bar{L}$, such that all the stable and unstable manifolds of (f, g) intersect transversally, and $J \in \mathcal{J}_{reg}(f, g)$. As in the transversal case we define the chain complex $CF_k(f, g, J, L, \mu; \Lambda)$ as the $\mathbb{Z}_2$ vector space consisting of formal sums of the form

$$\xi = \sum_{\substack{c \in \mathscr{C}_0 \\ I(c) = k}} \xi_c c$$

with $\mathbb{Z}_2$-coefficients ξ_c satisfying the finiteness condition

$$\#\{c : \xi_c \neq 0, E(c) \geq \kappa\} < \infty$$

for every $\kappa > 0$. The Novikov ring $\Lambda = \Lambda_\Gamma$ acts naturally on CF_*. Defining the boundary operator $\partial_k : CF_k \to CF_{k-1}$ in the usual way, we obtain homology groups

$$HF_k(f, g, J, L, \mu; \Lambda) := \frac{\ker\partial_{k+1}}{\mathrm{im}\partial_k}.$$

As in theorem A.17 one shows, that $HF_*(f, g, J, L, \mu; \Lambda)$ is canonically isomorphic to the moment Floer homology groups $HF_*(L, \mu; \Lambda)$.

To compute moment Floer homology we show that contribution of the cascades vanishes. To do that we endow the space of cascades with the structure of a space of Arnold-Givental type. Recall that $\dot{S} = dS(\mathrm{id})$ is the involution on the Lie algebra induced by the antisymplectic involution R. First note that R induces an involution R_* on the path space $\mathscr{P}$ by

$$R_*(x, \eta)(t) := (Rx, -\dot{S}(\eta))(1 - t).$$

One easily checks that if $c \in \tilde{\mathscr{C}_0}$ then c and R_*c represent the same element in $\mathscr{C}_0$. Choose now an almost complex structure $J \in \mathcal{J}$ which is independent of the t-variable and satisfies

$$R^*J = -J.$$

If (u, Ψ, Φ) is a cascade then one verifies that

$$R_*(u, \Psi, \Phi)(s, t) := (Ru, -\dot{S}(\Psi), \dot{S}(\Phi))(s, 1 - t)$$

is also a cascade. Moreover, one verifies that

$$\mathrm{ev}_j(R_*(u, \Psi, \Phi)) = R_*\mathrm{ev}_j(u, \Phi, \Psi), \quad j \in \{0, 1\}.$$

The following lemma shows us the relation between fixed gauge orbits of R_* and fixpoints of R_*.

Lemma 4.20 *Assume that (u, Ψ, Φ) is a cascade and $g \in \mathcal{G}_{loc}$ is a gauge transformation such that*

$$(u, \Psi, \Phi) = g_*(R_*(u, \Psi, \Phi)).$$

Then there exists $h \in \mathcal{G}_{loc}$ such that

$$h_*(u, \Psi, \Phi) = R_*(h_*(u, \Psi, \Phi)). \tag{25}$$

Proof: Choose $h \in \mathcal{G}_{loc}$ such that

$$h_*\Phi = 0, \quad \lim_{s\to\infty} h_*(u, \Psi)(s, t) = (p, 0)$$

where $p \in L \cap \mu^{-1}(0)$. Using the formula

$$h_*(u, \Psi, \Phi) = (hg(Sh)^{-1})_* \circ R_* \circ h_*(u, \Psi, \Phi).$$

we get

$$0 = h_*\Phi = (hg(Sh)^{-1})_*(0) = (hg(Sh)^{-1})^{-1}\partial_s(hg(Sh)^{-1})$$

and hence $hg(Sh)^{-1}$ is independent of s. Moreover, taking the limit $s \to \infty$ and recalling $p = Rp$ we have

$$(hg(Sh)^{-1})(t)p = p.$$

Since G acts freely on $\mu^{-1}(0)$ we obtain

$$hg(Sh)^{-1} \equiv \mathrm{id}$$

and hence h is the required gauge transformation, i.e. (25) holds. □

Proof of Theorem 4.16: In view of the Lemma 4.20 we can find in each fixed gauge orbit a fixed point. For fixed points we can define a sequence of involutions whose domain is the fixed point set of the previous one in the same way as for the pseudo-holomorphic disks in section 2. Using some equivariant version of Theorem 3.14 it follows that the space of cascades admits a Kuranishi structure of Arnold-Givental type. In particular, the only contribution to the boundary comes from the flow lines with zero cascades, i.e. the Morse-flow lines. This proves the theorem. □

A Morse-Bott theory

In this appendix we define a homology for Morse-Bott functions. Using an idea of Piunikhin, Salamon and Schwarz, see [PSS], we define Morse-Bott homology by counting flow lines with cascades. The homology is independent of the choice of the Morse-Bott function and hence isomorphic to the ordinary Morse homology.

A.1 Morse-Bott functions

Let (M, g) be a Riemannian manifold. A smooth function $f \in C^\infty(M, \mathbb{R})$ is called **Morse-Bott** if

$$\mathrm{crit}(f) := \{x \in M : df(x) = 0\}$$

is a submanifold of M and for each $x \in \mathrm{crit}(f)$ we have

$$T_x\mathrm{crit}(f) = \ker(\mathrm{Hess}(f)(x)).$$

Example A.1 *Let $M = \mathbb{R}^{k_0} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$. Write $x = (x_0, x_1, x_2)$ according to the splitting of M. Then*

$$f(x_0, x_1, x_2) = x_1^2 - x_2^2$$

is a Morse-Bott function on M.

Example A.2 *Let $M = \mathbb{R}$. Then*

$$f(x) = x^4$$

is no Morse-Bott function on M.

Theorem A.3 *Let (M, g) be a compact Riemannian manifold and f a Morse-Bott function on it. Let $y : \mathbb{R} \to M$ be a solution of*

$$\dot{y}(s) = -\nabla f(y(s)).$$

Then there exists $x \in \mathrm{crit}(f)$ and positive constants δ and c such that

$$\lim_{s\to\infty} y(s) = x, \quad |\dot{y}(s)| \le ce^{-\delta s}.$$

An analoguous result holds as s goes to $-\infty$.

Remark A.4 *Without the Morse-Bott condition Theorem A.3 will in general not hold. Let M and f be as in Example A.2. Then*

$$y(s) := \frac{1}{\sqrt{8s}}$$

is a solution of the gradient equation which converges to the critical point 0 as s goes to ∞. But the convergence is not exponential.

Proof of Theorem A.3: Since M is compact and $\mathrm{crit}(f)$ is normally hyperbolic there exists $x \in \mathrm{crit}(f)$ such that

$$\lim_{s\to\infty} y(s) = x.$$

Set

$$A(s) := f(y(s)) - f(x).$$

Then for some $\epsilon > 0$ we have

$$\dot{A}(s) = -|\dot{y}(s)|^2 = -|\nabla f(y(s))|^2 \leq -\epsilon A(s).$$

The last inequality follows from the Morse-Bott assumption. Hence there exists a constant $c_0 > 0$ such that

$$A(s) \leq c_0 e^{-\epsilon s}.$$

This proves the theorem. $\square$

A.2 Flow lines with cascades

Let (M, g) be a compact Riemannian manifold, f a Morse-Bott function on M, g_0 a Riemannian metric on $\mathrm{crit}(f)$, and h a Morse-function on $\mathrm{crit}(f)$. We assume that h satisfies the Morse-Smale condition, i.e. stable and unstable manifolds intersect transversally. For a critical point c on h let $\mathrm{ind}_f(c)$ be the number of negative eigenvalues of $\mathrm{Hess}(f)(c)$ and $\mathrm{ind}_h(c)$ be the number of negative eigenvalues of $\mathrm{Hess}(h)(c)$. We define

$$\mathrm{Ind}(c) := \mathrm{Ind}_{f,h}(c) := \mathrm{ind}_f(c) + \mathrm{ind}_h(c).$$

Definition A.5 *For $c_1, c_2 \in \mathrm{crit}(h)$, and $m \in \mathbb{N}$ a* **flow line from c_1 to c_2 with m cascades**

$$(\boldsymbol{x}, \boldsymbol{T}) = ((x_k)_{1\leq k\leq m}, (t_k)_{1\leq k\leq m-1})$$

consists of $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_\geq := \{r \in \mathbb{R} : r \geq 0\}$ which satisfy the following conditions:

(i) *$x_k \in C^\infty(\mathbb{R}, M)$ are nonconstant solutions of*

$$\dot{x}_k = -\nabla f(x_k).$$

(ii) *There exists $p \in W^u_h(c_1) \subset \mathrm{crit}(f)$ and $q \in W^s_h(c_2)$ such that $\lim_{s\to-\infty} x_1(s) = p$ and $\lim_{s\to\infty} x_m(s) = q$.*

(iii) *for $1 \leq k \leq m-1$ there are Morse-flow lines $y_k \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ of h, i.e. solutions of*

$$\dot{y}_k = -\nabla h(y_k),$$

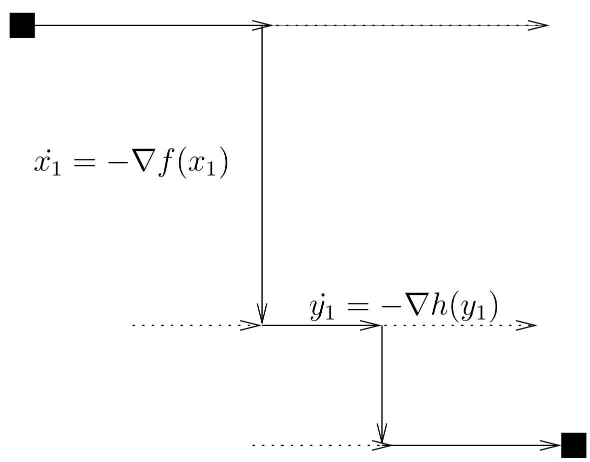
such that

$$\lim_{s\to\infty} x_k(s) = y_k(0)$$

and

$$\lim_{s\to-\infty} x_{k+1}(s) = y_k(t_k).$$

A **flow line with zero cascades from** c_1 **to** c_2 *is an ordinary Morse flow line of* h *on* $\mathrm{crit}(f)$ *from* c_1 *to* c_2*.*



A flow line with cascades

Remark A.6 *In Definition A.5 we do not require that the Morse-flow lines are nonconstant. It may happen that a cascade converges to a critical point of* h*, but the flow line will only remain for a finite time on the critical point.*

We denote the space of flow lines with m cascades from c_1 to $c_2 \in \mathrm{crit}(h)$ by

$$\tilde{\mathcal{M}}_m(c_1, c_2).$$

The group $\mathbb{R}$ acts by timeshift on the set of solutions connecting two critical points on the same level $\tilde{\mathcal{M}}_0(c_1, c_2)$ and the group $\mathbb{R}^m$ acts on $\tilde{\mathcal{M}}_m(c_1, c_2)$ by timeshift on each cascade, i.e.

$$x_k(s) \mapsto x_k(s + s_k).$$

We denote the quotient by

$$\mathcal{M}_m(c_1, c_2).$$

We define the **set of flow lines with cascades from** c_1 **to** c_2 by

$$\mathcal{M}(c_1, c_2) := \bigcup_{m\in\mathbb{N}_0} \mathcal{M}_m(c_1, c_2).$$

Immediately from the gradient equation the following lemma follows.

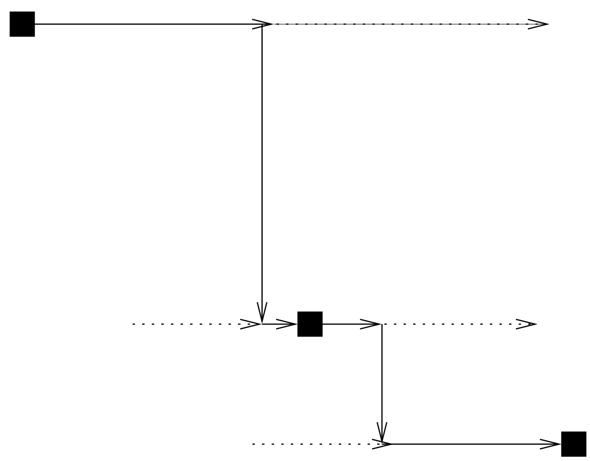
Lemma A.7 *If $f(c_1) < f(c_2)$ then $\mathcal{M}(c_1, c_2)$ is empty. If $f(c_1) = f(c_2)$ then $\mathcal{M}(c_1, c_2)$ contains only flow lines with zero cascades. If $f(c_1) > f(c_2)$ then $\mathcal{M}(c_1, c_2)$ contains no flow line with zero cascades.*

A sequence of flow lines with cascades may break up in the limit into a connected chain of flow lines with cascades. To deal with this phenomenon, we make the following definitions.

Definition A.8 *Let $c, d \in \mathrm{crit}(h)$. A* **broken flow line with cascades** *from c to d*

$$\boldsymbol{v} = (v_j)_{1 \leq j \leq \ell}$$

for $\ell \in \mathbb{N}$ consists of flow lines with cascades v_j from c_{j-1} to $c_j \in \mathrm{crit}(h)$ for $0 \leq j \leq \ell$ such that $c_0 = c$ and $c_\ell = d$.



A broken flow line with cascades

Definition A.9 *Assume that $c, d \in \mathrm{crit}(h)$. Suppose that v^ν for $\nu \in \mathbb{N}$ is a sequence of flow lines with cascades which satisfies the following condition. There exists $\nu_0 \in \mathbb{N}$ such that for every $\nu \geq \nu_0$ it holds that v^ν is a flow line with cascades from c to d. There are two cases. In the first case c and d lie on the same level and hence $v^\nu \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ is a flow line with zero cascades for every $\nu \geq \nu_0$, in the second case c and d lie on different levels and hence $v^\nu = ((x_k^\nu)_{1 \leq k \leq m^\nu}, (t_k)_{1 \leq k \leq m^\nu - 1})$ is a flow line with at least one cascade for every $\nu \geq \nu_0$. We say that v^ν* **Floer-Gromov converges** *to a broken flow line with cascades $\boldsymbol{v} = (v_j)_{1 \leq j \leq \ell}$ from c to d if the following holds.*

(a) *In the first case, where the v^ν's are flow lines with zero cascades for large enough ν's, all v_j's are flow lines with zero cascades and there exists real numbers s_j^ν for $\nu \geq \nu_0$ such that $(s_j^\nu)_*(v^\nu)(\cdot) := v^\nu(\cdot + s_j^\nu)$ converges in the C^∞_{loc}-topology to v_j.*

(b) *In the second case, where the v^ν's have at least one cascade for ν large enough, we require the following conditions.*

(i) *If $v_j \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ is a flow line with zero cascades, then there exists a sequence of solutions $y_j^\nu \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ of $\dot{y}_j^\nu = -\nabla h(y_j^\nu)$ converging in C^∞_{loc} to v_j, a sequence of real numbers s_j^ν, and a sequence of integers $k^\nu \in [1, m^\nu]$ such that either $\lim_{s\to-\infty} x^\nu_{k^\nu}(s) = y_j^\nu(s_j^\nu)$ or $\lim_{s\to\infty} x^\nu_{k^\nu}(s) = y_j^\nu(s_j^\nu)$.*

(ii) *If v_j is a flow line with at least one cascade, then we write $v_j = ((x_{i,j})_{1\leq i\leq m_j}, (t_{i,j})_{1\leq i\leq m_j-1}) \in \tilde{\mathcal{M}}_{m_j}(c_{j-1}, c_j)$ for $m_j \geq 1$. We require that there exist surjective maps $\gamma^\nu \colon [1, \sum_{p=1}^\ell m_p] \to [1, m^\nu]$, which are monotone increasing, i.e. $\gamma^\nu(\lambda_1) \leq \gamma^\nu(\lambda_2)$ for $\lambda_1 \leq \lambda_2$, and real numbers s^ν_λ for every $\lambda \in [1, \sum_{p=1}^\ell m_p]$, such that*

$$(s^\nu_\lambda)_* x^\nu_{\gamma^\nu(\lambda)}(\cdot) = x^\nu_{\gamma^\nu(\lambda)}(\cdot + s^\nu_\lambda) \xrightarrow{C^\infty_{loc}} x_\lambda$$

where $x_\lambda = x_{i,j}$ such that $\lambda = \sum_{p=1}^j m_p + i$. For $\lambda \in [1, \sum_{p=1}^\ell m_p - 1]$ we set

$$\tau_\lambda = \begin{cases} t_{i,j} & \lambda = \sum_{p=1}^j m_p + i, \quad 0 < i < m_{j+1} \\ \infty & \lambda = \sum_{p=1}^j m_p \end{cases}$$

and

$$\tau^\nu_\lambda = \begin{cases} t^\nu_{\gamma^\nu(\lambda)} & \lambda = \max\{\lambda' \in [1, \sum_{p=1}^\ell m_p - 1] : \gamma^\nu(\lambda') = \gamma^\nu(\lambda)\} \\ 0 & \text{otherwise.} \end{cases}$$

Now we require, that

$$\lim_{\nu\to\infty} \tau^\nu_\lambda = \tau_\lambda.$$

Here we use the convention that a sequence of real numbers τ^ν converges to infinity, if for every $n \in \mathbb{N}$ there exists a $\nu_0(n) \in \mathbb{N}$ such that $\tau^\nu \geq n$ for $\nu \geq \nu_0(n)$.

Theorem A.10 (Compactness) *Let v^ν be a sequence of flow lines with cascades. Then there exists a subsequence v^{ν_j} and a broken flow line with cascades $\boldsymbol{v}$ such that v^{ν_j} Floer-Gromov converges to $\boldsymbol{v}$.*

Proof: First assume that there exists a subsequence ν_j such that v^{ν_j} are flow lines with zero cascades. In this case Floer-Gromov convergence to a broken flow line without cascades follows from the classical case, see [Sch1, Proposition 2.35]. Otherwise pick a subsequence ν_j of ν such that v^{ν_j} are flow lines with at least one cascade. Since the number of critical points of h is finite, we can perhaps after passing over to a further subsequence assume that all the v^{ν_j} are flow lines with cascades from a common critical point c of h to a common critical point d of h. Since the number of connected components of $\mathrm{crit}(f)$ is finite, we

can assume by passing to a further subsequence that the number of cascades $m^\nu = m$ is independent on ν.
For simplicity of notation we denote the subsequence ν_j again by ν. We consider the sequence of points $p^\nu := \lim_{s\to\infty} x_1^\nu(s) \in \mathrm{crit}(f)$. Let $y^\nu \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ be the unique solution of the problem

$$\dot{y}^\nu(s) = -\nabla h(y^\nu(s)), \quad y^\nu(0) = p^\nu.$$

Note that for every ν

$$\lim_{s\to-\infty} y^\nu(s) = c.$$

Using again [Sch1, Proposition 2.35] it follows that perhaps passing over to a subsequence (also denoted by ν) there exists $p \in \mathrm{crit}(f)$, a nonnegative integer $m_0 \in \mathbb{N}$, and a sequence of Morse-flow lines $v_j \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ with respect to the gradient flow of h for $1 \le j \le m_0$ such that the following holds.

(i) $\lim_{\nu\to\infty} p^\nu = p$,

(ii) If $m_0 = 0$, then $p \in W_h^u(c)$,

(iii) If $m_0 \ge 1$, then

$$\begin{aligned} \lim_{s\to-\infty} v_1(s) &= c, \\ \lim_{s\to\infty} v_j(s) &= \lim_{s\to-\infty} v_{j+1}(s), \quad 1 \le j \le m_0 - 1, \\ p &\in W_h^u(\lim_{s\to\infty} v_{m_0}(s)). \end{aligned}$$

Using induction on $\mu \in [1, m]$, where m is the number of cascades of each v^ν, the following claim follows as [Sch1, Proposition 2.35].

Claim: *There exist a subsequence of ν (still denoted by ν), a nonnegative integer $\ell_\mu^1 \in \mathbb{N}_0$, a positive integer $\ell_\mu^2 \in \mathbb{N}$, a broken flow line with cascades $\boldsymbol{v}_\mu = (v_j)_{1\le j\le \ell_\mu^1}$ from c to some critical point c_μ of h, a sequence of cascades $x_k^\mu \in C^\infty(\mathbb{R}, M)$ for $1 \le k \le \ell_\mu^2$, i.e. of nonconstant solutions of the ordinary differential equation $\dot{x}_k^\mu = -\nabla f(x_k^\mu)$, and a sequence of nonnegative real number t_k^μ for $1 \le k \le \ell_\mu^2 - 1$ such that the following conditions are satisfied.*

(i) *If $\ell_\mu^1 = 0$, then $\lim_{s\to-\infty} x_1(s) \in W_h^u(c)$, otherwise $\lim_{s\to-\infty} x_1(s) \in W_h^u(c_\mu)$,*

(ii) *For $1 \le k \le \ell_\mu^2 - 1$ there are Morse-flow lines $y_k^\mu \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ of h, such that*

$$\lim_{s\to\infty} x_k^\mu(s) = y_k^\mu(0), \quad \lim_{s\to-\infty} x_{k+1}^\mu(s) = y_k^\mu(t_k^\mu),$$

(iii) *If $v_j \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ is a flow line with zero cascades, then there exists a sequence of solutions $y_j^\nu \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ of $\dot{y}_j^\nu = -\nabla h(y_j^\nu)$ converging in C^∞_{loc} to v_j, a sequence of real numbers s_j^ν, and an integer $k \in [1, \mu]$ such that $\lim_{s\to-\infty} x_k^\nu(s) = y_j^\nu(s_j^\nu)$.*

(iv) *If v_j is a flow line with at least one cascade, then we are able to write $v_j = ((x_{i,j})_{1\le i\le m_j}, (t_{i,j})_{1\le i\le m_j-1}) \in \tilde{\mathcal{M}}_{m_j}(c_{j-1}, c_j)$ for $m_j \ge 1$. We require that there exists a surjective map $\gamma_\mu : [1, \sum_{p=1}^{\ell_\mu^1} m_p + \ell_\mu^2] \to [1, \mu]$, which is monotone increasing, i.e. $\gamma_\mu(\lambda_1) \le \gamma_\mu(\lambda_2)$ for $\lambda_1 \le \lambda_2$, and real numbers s_λ^ν for every $\lambda \in [1, \sum_{p=1}^{\ell_\mu^1} m_p + \ell_\mu^2]$, such that*

$$(s_\lambda^\nu)_* x^\nu_{\gamma^\nu(\lambda)}(\cdot) = x^\nu_{\gamma^\nu(\lambda)}(\cdot + s_\lambda^\nu) \xrightarrow{C^\infty_{loc}} x_\lambda$$

where $x_\lambda = x_{i,j}$ if $\lambda = \sum_{p=1}^{j} m_p + i$ for $0 \le j \le \ell_\mu^1$ and $0 \le i \le m_j - 1$, or $x_\lambda = x^\mu_{\lambda - \sum_{p=1}^{\ell_\mu^1} m_p}$ for $\lambda \in [\sum_{p=1}^{\ell_\mu^1} m_p + 1, \sum_{p=1}^{\ell_\mu^1} m_p + \ell_\mu^2]$. For $\lambda \in [1, \sum_{p=1}^{\ell_\mu^1} m_p + \ell_\mu^2 - 1]$ we set

$$\tau_\lambda = \begin{cases} t_{i,j} & \lambda = \sum_{p=1}^{j} m_p + i, \quad 0 < i < m_{j+1} \\ \infty & \lambda = \sum_{p=1}^{j} m_p \\ t^\mu_{\lambda - \sum_{p=1}^{\ell_\mu^1} m_p} & \lambda \in [\sum_{p=1}^{\ell_\mu^1} m_p + 1, \sum_{p=1}^{\ell_\mu^2} m_p + \ell_\mu^2] \end{cases}$$

and

$$\tau_\lambda^\nu = \begin{cases} t^\nu_{\gamma^\nu(\lambda)} & \lambda = \max\{\lambda' \in [1, \sum_{p=1}^{\ell_\mu^1} m_p + \ell_\mu^2 - 1] : \gamma^\nu(\lambda') = \gamma^\nu(\lambda)\} \\ 0 & \text{otherwise.} \end{cases}$$

We require, that

$$\lim_{\nu\to\infty} \tau_\lambda^\nu = \tau_\lambda.$$

(v) $\lim_{\nu\to\infty} \lim_{s\to\infty} x_\mu^\nu(s)$ *exists and equals* $\lim_{s\to\infty} x^\mu_{\ell_\mu^2}(s)$.

Given the claim for $\mu = m$, the Theorem follows now by applying [Sch1, Proposition 2.35] again. $\square$

Theorem A.11 *Let $c_1, c_2 \in \mathrm{crit}(h)$. For generic choice of the Riemannian metric g of M the space $\mathcal{M}(c_1, c_2)$ is a smooth finite dimensional manifold. Its dimension is given by*

$$\dim \mathcal{M}(c_1, c_2) = \mathrm{ind}(c_1) - \mathrm{ind}(c_2) - 1.$$

If $\mathrm{Ind}(c_1) - \mathrm{Ind}(c_2) = 1$ then $\mathcal{M}(c_1, c_2)$ is compact and hence a finite set.

The rest of this subsection is devoted to the proof of Theorem A.11. Choose $0 < \delta < \min(\{|\lambda| : \lambda \in \sigma(\mathrm{Hess}(f)(x)) \setminus \{0\}, x \in \mathrm{crit}(f)\}$. For a smooth cutoff function β such that $\beta(s) = -1$ if $s < 0$ and $\beta(s) = 1$ if $s > 1$ define

$$\gamma_\delta : \mathbb{R} \to \mathbb{R}, \quad s \mapsto e^{\delta\beta(s)s}.$$

Let Ω be an open subset of $\mathbb{R}$. We define the $|| \, ||_{k,p,\delta}$-norm for a locally integrable function $f : \Omega \to \mathbb{R}$ with weak derivatives up to order k by

$$||f||_{k,p,\delta} := \sum_{i=0}^{k} ||\gamma_\delta \partial^i f||_p.$$

We denote

$$W^{k,p}_\delta(\Omega) := \{f \in W^{k,p}(\Omega) : ||f||_{k,p,\delta} < \infty\} = \{f \in W^{k,p}(\Omega) : \gamma_\delta f \in W^{k,p}(\Omega)\}.$$

We also set

$$L^p_\delta(\Omega) := W^{0,p}_\delta(\Omega).$$

Let

$$T_h(t) \in \mathrm{Diff}(\mathrm{crit}(f))$$

be the smooth family of diffeomorphisms which assigns to $p \in \mathrm{crit}(f)$ the point $x_p(t)$ where x_p is the unique flow line of h with $x_p(0) = p$. We define

$$\mathcal{B} := \mathcal{B}^{1,p}_\delta(M, f, h)$$

as the Banach manifold consisting of all tuples $v = ((x_j)_{1 \le j \le m}, (t_j)_{1 \le j \le m-1}) \in (W^{1,p}_{loc}(\mathbb{R}, M))^m \times \mathbb{R}^{m-1}_+$ where $\mathbb{R}_+ := \{r \in \mathbb{R} : r > 0\}$ and $m \in \mathbb{N}$ which satisfy the following conditions:

(Asympotic behaviour) For $1 \le j \le m$ there exists $p_j, q_j \in \mathrm{crit}(f)$, $\xi_{1,j} \in W^{1,p}_\delta((-\infty, T], T_{p_j}M)$, $\xi_{2,j} \in W^{1,p}_\delta([T, \infty), T_{q_j}M)$ for $T \in \mathbb{R}$ such that

$$x_j(s) = \exp_{p_j}(\xi_{1,j}(s)), \quad s \le -T, \qquad x_j(s) = \exp_{q_j}(\xi_{2,j}(s)), \quad s \ge T,$$

where exp is taken with respect to the Riemannian metric g of M.

(Connectedness) $p_{j+1} = T_h(t_j) q_j$ for $1 \le j \le m-1$.

To define local charts on $\mathcal{B}$ choose $v = ((x_j)_{1 \le j \le m}, (t_j)_{1 \le j \le m-1}) \in \mathcal{B}$ such that all the x_j for $1 \le j \le m$ are smooth and define a neighbourhood of v in $\mathcal{B}$ via the exponential map of g [8]. There are two natural smooth evaluation maps

$$\mathrm{ev}_1, \mathrm{ev}_2 : \mathcal{B} \to \mathrm{crit}(f), \qquad \mathrm{ev}_1(v) = p_1, \quad \mathrm{ev}_2(v) = q_m.$$

After choosing cutoff functions and smooth trivializations χ_{p_j} and χ_{q_j} of TM near p_j respectively q_j the tangent space of $\mathcal{B}$ at v can naturally be identified with tuples

$$\zeta = ((\xi_{j,0}, \xi_{j,1}, \xi_{j,2})_{1 \le j \le m}, (\tau_j)_{1 \le j \le m-1}) \in$$
$$\bigoplus_{j=1}^{m} (W^{1,p}_\delta(\mathbb{R}, x_j^* TM) \times T_{p_j}\mathrm{crit}(f) \times T_{q_j}\mathrm{crit}(f)) \times \mathbb{R}^{m-1}$$

[8] Note that the differentiable structure of $\mathcal{B}$ is independent of the metric g on M

which satisfy

$$dT_h(t_j)\xi_{j,2} + \frac{d}{dt}(T_h(0)q_j)\tau_j = \xi_{j+1,1} \quad 1 \le j \le m-1. \tag{26}$$

$T_v\mathcal{B}$ is a Banach space with norm

$$||\zeta|| := \sum_{j=1}^{m}(||\xi_{j,0}||_{1,p,\delta} + ||\xi_{j,1}|| + ||\xi_{j,2}||) + \sum_{j=1}^{m-1}|\tau_j| \tag{27}$$

Let $\mathcal{E}$ be the Banachbundle over $\mathcal{B}$ whose fiber at $v \in \mathcal{B}$ is given by

$$\mathcal{E}_v := \bigoplus_{j=1}^{m} L^p_\delta(\mathbb{R}, x_j^*TM).$$

Set

$$\tilde{\mathcal{M}} := \mathcal{F}^{-1}(0)$$

where

$$\mathcal{F} : \mathcal{B} \to \mathcal{E}, \quad v \mapsto (\dot{x}_k + \nabla f(x_k))_{1 \le k \le m}.$$

Where $\nabla = \nabla_g$ is the Levi-Civita connection of the Riemannian metric g on M. Note that $\mathcal{F} = \mathcal{F}_g$ depends on g. Let

$$D_v := D\mathcal{F}(v) : T_v\mathcal{B} \to \mathcal{E}_v$$

be the vertical differential of $\mathcal{F}$ at $v \in \mathcal{F}^{-1}(0)$. If $p \in \operatorname{crit}(f)$ denote by $\dim_p \operatorname{crit}(f)$ the local dimension of $\operatorname{crit}(f)$ at p. Note that it follows from the Morse-Bott condition that $\dim_p \operatorname{crit}(f)$ equals the dimension of the kernel of $\operatorname{Hess}(f)(p)$. Then we have

Lemma A.12 *D_v is a Fredholm-operator of index*

$$\operatorname{ind}(D_v) = \operatorname{ind}_f(\operatorname{ev}_1(v)) + \dim_{\operatorname{ev}_1(v)} \operatorname{crit}(f) - \operatorname{ind}_f(\operatorname{ev}_2(v)) + m - 1,$$

where $m = m(v)$ is the number of cascades.

Proof: For $1 \le j \le m$ let

$$D_{v,j} : W^{1,p}_\delta(\mathbb{R}, x_j^*TM) \to L^p_\delta(\mathbb{R}, x_j^*TM)$$

be the restriction of D_v to $W^{1,p}_\delta(\mathbb{R}, x_j^*TM)$. It suffices to show, that $D_{v,j}$ is a Fredholm-operator of index

$$\operatorname{ind}(D_{v,j}) = \operatorname{ind}_f(p_j) - \operatorname{ind}_f(q_j) - \dim_{q_j} \operatorname{crit}(f).$$

Write

$$D_{v,j} := \partial_s + A(s).$$

Then

$$A_1 := \lim_{s \to -\infty} A(s) = \mathrm{Hess}(f)(p_j), \quad A_2 := \lim_{s \to \infty} A(s) = \mathrm{Hess}(f)(q_j).$$

Define the continuous isomorphisms

$$\phi_1 : L^p_\delta \to L^p, \quad f \mapsto f\gamma_\delta, \qquad \phi_2 : W^{1,p}_\delta \to W^{1,p}, \quad f \mapsto f\gamma_\delta.$$

Define

$$\tilde{D}_v : W^{1,p}(\mathbb{R}, x_j^*TM) \to L^p(\mathbb{R}, x_j^*TM)$$

by

$$\tilde{D}_v := \phi_1 D_v \phi_2^{-1}.$$

D_v is exactly a Fredholm-operator if $\tilde{D}_v$ is a Fredholm-operator. In this case

$$\mathrm{ind}(D_v) = \mathrm{ind}(\tilde{D}_v).$$

For $\xi \in W^{1,p}(\mathbb{R}, x_j^*TM)$ we calculate

$$\begin{aligned}
\tilde{D}_v\xi &= \phi_1 D_v \phi_2^{-1}\xi \\
&= \phi_1 D_v(\xi\gamma_{-\delta}) \\
&= \phi_1(\partial_s(\xi\gamma_{-\delta}) + A(s)\xi\gamma_{-\delta}) \\
&= \phi_1((\partial_s\xi)\gamma_{-\delta} + (\delta\beta(s) + \delta\partial_s\beta(s)s)\xi\gamma_{-\delta} + A(s)\xi\gamma_{-\delta}) \\
&= \partial_s\xi + (A(s) + \delta(\beta(s) + \partial_s\beta(s)s)\mathrm{id})\xi.
\end{aligned}$$

Hence $\tilde{D}_v$ is given by

$$\tilde{D}_v = \partial_s + B(s)$$

where

$$B(s) = A(s) + \delta(\beta(s) + \partial_s\beta(s)s)\mathrm{id}.$$

Let

$$B_j := A_j + (-1)^j\delta, \quad j \in \{1, 2\}.$$

Then

$$\lim_{s \to -\infty} B(s) = B_1, \quad \lim_{s \to \infty} B(s) = B_2$$

and the B_j are invertible by our choice of δ. If $p = 2$ then it follows from [RS1] that $\tilde{D}_v$ is a Fredholm-operator of the required index. For general p the lemma follows from [S2]. See also [Sch2], for an alternative proof. □

For $n \in \mathbb{N}$ we define the evaluation maps

$$\mathrm{EV}_n : \tilde{\mathcal{M}}^n \to \mathrm{crit}(f)^n \times \mathrm{crit}(f)^n \cong \mathrm{crit}(f)^{2n},$$

$$(v_1, \ldots, v_n) \mapsto (\mathrm{ev}_1(v_j)_{1 \le j \le n}, \mathrm{ev}_2(v_j)_{1 \le j \le n}),$$

where $\tilde{\mathcal{M}} = \mathcal{F}^{-1}(0)$ as introduced above. For critical points c_1, c_2 of h we define $A_n(c_1, c_2)$ to be the submanifold of $\mathrm{crit}(f)^n \times \mathrm{crit}(f)^n$ consisting of all tuples $((p_j)_{1 \le j \le n}, (q_j)_{1 \le j \le n}) \in \mathrm{crit}(f)^n \times \mathrm{crit}(f)^n$ such that $p_1 \in W^u_h(c_1)$, $q_n \in W^s_h(c_2)$, and $p_{j+1} = q_j$ for $1 \le j \le n-1$. We shall prove the following theorem.

Theorem A.13 *For generic Riemannian metric g on M the set $\tilde{\mathcal{M}}$ has the structure of a finite dimensional manifold. Its local dimension*

$$\dim_v \tilde{\mathcal{M}} = \mathrm{ind} D_v = \mathrm{ind}_f(\mathrm{ev}_1(v)) + \dim_{\mathrm{ev}_1(v)} \mathrm{crit}(f) - \mathrm{ind}_f(\mathrm{ev}_2(v)) + m - 1, \quad (28)$$

where $m = m(v)$ is the number of cascades. Moreover, for every $n \in \mathbb{N}$ and for every $c_1, c_2 \in \mathrm{crit}(h)$ the evaluation maps EV_n intersects $A_n(c_1, c_2)$ transversally.

Definition A.14 *Assume that f is Morse-Bott function on a compact manifold M, h is a Morse-function on $\mathrm{crit}(f)$, and g_0 is a Riemannian metric on $\mathrm{crit}(f)$, such that h and g_0 satisfy the Morse-Smale condition, i.e. stable and unstable manifolds of the gradient flow of h with respect to g_0 on $\mathrm{crit}(f)$ intersect transversally. We say that a Riemannian metric g on M is* (f, h, g_0)**-regular** *if it satisfies the conditions of Theorem A.13.*

Proof of Theorem A.13: For a positive integer ℓ let $\mathcal{R}^\ell$ be the Banach manifold of Riemannian metrics on M of class C^ℓ. Let

$$\mathcal{F} : \mathcal{R}^\ell \times \mathcal{B} \to \mathcal{E}$$

be defined by

$$\mathcal{F}(g, v) = (\dot{x}_k + \nabla_g f(x_k))_{1 \le k \le m} = (\dot{x}_k + g^{-1} df(x_k))_{1 \le k \le m},$$

where $g^{-1} \colon T^*M \to TM$ is defined as the inverse of the map defined by

$$g(\nabla_g f, \cdot) = df(\cdot).$$

We will prove that the universal moduli space

$$\mathcal{U}^\ell := \{(v, g) \in \mathcal{B} \times \mathcal{R}^\ell : \mathcal{F}(v, g) = 0\}$$

is a separable manifold of class C^ℓ. To show that, we have to verify that

$$D_{v,g} : T_v\mathcal{B} \times T_g\mathcal{R}^\ell \to \mathcal{E}_v$$

given by

$$\begin{aligned} D_{v,g}(\zeta, A) &= (\partial_s \zeta_k + \nabla_{\zeta_k} \nabla f(x_k))_{1 \le k \le m} - (g^{-1} A g^{-1} df(x_k))_{1 \le k \le m} \\ &= D_v \zeta - (g^{-1} A g^{-1} df(x_k))_{1 \le k \le m} \end{aligned}$$

is onto for every $(v, g) \in \mathcal{U}^\ell$. Here ∇_{ζ_k} is the Levi-Civita connection of the metric g.

The tangent space of $\mathcal{R}^\ell$ at $g \in \mathcal{R}^\ell$ consists of all symmetric C^ℓ-sections from M to $T^*M \times T^*M$. Since D_v is a Fredholm operator, it has a closed range and a finite dimensional cokernel. Hence $D_{v,g}$ has a closed range and a finite dimensional cokernel and it only remains to prove that its range is dense. To see this, let

$$\eta \in (\mathcal{E}_v)^* = \bigoplus_{j=1}^{m} L^q_{-\delta}(\mathbb{R}, x_j^* TM), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

such that η vanishes on the range of $D_{v,g}$, i.e.

$$\sum_{j=1}^{m} \int_{\mathbb{R}} \langle \eta_j, (D_v\zeta)_j \rangle ds = 0 \tag{29}$$

for every $\zeta \in T_v\mathcal{B}$ and

$$\sum_{j=1}^{m} \int_{\mathbb{R}} \langle \eta_j, g^{-1}Ag^{-1}df(x_j) \rangle ds = 0 \tag{30}$$

for every $A \in T_g\mathcal{R}^\ell$. It follows from (29) that η_j is continuously differentiable for $1 \leq j \leq m$. Now (30) implies that η vanishes identically. This proves that $D_{v,g}$ is onto for every $(v,g) \in \mathcal{U}^\ell$. Now it follows from the implicit function theorem that $\mathcal{U}^\ell$ is a Banach-manifold.

The differential $d\pi^\ell$ of the projection

$$\pi^\ell : \mathcal{U}^\ell \to \mathcal{R}^\ell, \quad (v,g) \mapsto g$$

at a point $(v,g) \in \mathcal{U}^\ell$ is just the projection

$$d\pi^\ell(v,g) \colon T_{(v,g)}\mathcal{U}^\ell \to T_g\mathcal{R}^\ell, \quad (\zeta, A) \mapsto A.$$

The kernel of $d\pi^\ell(v,g)$ is isomorphic to the kernel of D_v. Its image consists of all A such that $(g^{-1}Ag^{-1}df(x_k))_{1\leq k\leq m} \in \mathrm{im}D_v$. Moreover, $\mathrm{im}d\pi^\ell(v,g)$ is a closed subspace of $T_g\mathcal{R}^\ell$, and, since $D_{v,g}$ is onto, it has the same (finite) codimension as the image of D_v. Hence $d\pi^\ell(v,g)$ is a Fredholm operator of the same index as $D_{v,g}$. In particular, the projection π^ℓ is a Fredholm map and it follows from the Sard-Smale theorem that for ℓ sufficiently large, the set $\mathcal{R}^\ell_{reg}$ of regular values of π^ℓ is dense in $\mathcal{R}^\ell$. Note that $g \in \mathcal{R}^\ell$ is a regular value of π^ℓ exactly if D_v is surjective for every $v \in \mathcal{F}_g^{-1}(0)$. Here $\mathcal{F}_g = \mathcal{F}(\cdot, g)$.

For $c > 0$ let $\mathcal{U}^{c,\ell} \subset \mathcal{U}^\ell$ be the set of pairs $(v,g) \in \mathcal{U}^\ell$ such that

$$||\partial_s x_j(s)|| \leq ce^{-|s|/c}, \quad 1 \leq j \leq m, \qquad \frac{1}{c} \leq t_k \leq c, \quad 1 \leq j \leq m-1.$$

The space

$$\tilde{\mathcal{M}}^{\ell,c}(g) := \{v : (v,g) \in \mathcal{U}^{\ell,c}\}$$

is compact for every g. Indeed, the uniform exponential decay prevents the cascades from breaking up into several pieces. It follows that the set $\mathcal{R}^{\ell,c}_{reg}$ consisting of all $g \in \mathcal{R}^\ell$ such that D_v is onto for all $(v,g) \in \tilde{\mathcal{M}}^{\ell,c}(g)$ is open and dense in $\mathcal{R}^\ell$. Hence the set

$$\mathcal{R}^{\infty,c}_{reg} := \mathcal{R}^{\ell,c}_{reg} \cap \mathcal{R}$$

is dense in $\mathcal{R}^\ell$ with respect to the C^ℓ-topology. Here $\mathcal{R} = \mathcal{R}^\infty$ denotes the Fréchet manifold of smooth metrics on M. Since this holds for every ℓ if follows

that $\mathcal{R}^{\infty,c}$ is dense in $\mathcal{R}$ with respect to the C^∞-topology. Using compactness again one obtains that $\mathcal{R}^c_{reg}$ is also C^∞-open. It follows that the set

$$\mathcal{R}_{reg} := \bigcap_{c\in\mathbb{N}} \mathcal{R}^{\infty,c}_{reg}$$

is a countable intersection of open and dense subsets of $\mathcal{R}$.

To prove that the evaluation maps EV_n intersect $A_n(c_1,c_2)$ transversally for generic g, we show that the evaluation maps $\mathrm{ev}_j : \mathcal{U}^\ell \to \mathrm{crit}(f)$ for $j\in\{1,2\}$ are submersive. Let $(v,g)\in\mathcal{U}^\ell$ and $\xi\in T_{\mathrm{ev}_j(v,g)}\mathrm{crit}(f)$. We have to show that there exists $(\zeta,A)\in\ker D_{v,g}$ such that

$$d(\mathrm{ev}_j)(v,g)(\zeta,A)=\xi. \tag{31}$$

Choose some arbitrary $(\zeta_0,A_0)\in T_v\mathcal{B}\times T_g\mathcal{R}^\ell$ such that

$$d(\mathrm{ev}_j)(v,g)(\zeta_0,A_0)=\xi.$$

In the same way as one proved that $D_{v,g}$ is surjective one can also show that already $D_{v,g}$ restricted to $\{\zeta\in T_v\mathcal{B} : d(\mathrm{ev}_j)\zeta=0\}\times T_g\mathcal{R}^\ell$ is surjective. Hence there exist $(\zeta_1,A_1)\in T_v\mathcal{B}\times T_g\mathcal{R}^\ell$ such that

$$D_{v,g}(\zeta_0,A_0)=D_{v,g}(\zeta_1,A_1)$$

and

$$d(\mathrm{ev}_j)(v,g)(\zeta_1,A_1)=0.$$

Now set

$$(\zeta,A):=(\zeta_0-\zeta_1,A_0-A_1).$$

Then (ζ,A) lies in the kernel of $D_{v,g}$ and satisfies (31). This proves the theorem.
□

For $c_1,c_2\in\mathrm{crit}(h)\subset\mathrm{crit}(f)$ define

$$\begin{aligned}
\tilde{\mathcal{M}}^-(c_1) &:= \{v\in\tilde{\mathcal{M}} : \mathrm{ev}_1(v)\in W^u_h(c_1)\},\\
\tilde{\mathcal{M}}^+(c_2) &:= \{v\in\tilde{\mathcal{M}} : \mathrm{ev}_2(v)\in W^s_h(c_2)\},\\
\tilde{\mathcal{M}}(c_1,c_2) &:= \tilde{\mathcal{M}}^-(c_1)\cap\tilde{\mathcal{M}}^+(c_2).
\end{aligned}$$

Immediately from Theorem A.13 the following Corollary follows.

Corollary A.15 *For generic Riemannian metric g on M the spaces $\tilde{\mathcal{M}}_u(c_1)$, $\tilde{\mathcal{M}}_s(c_2)$, and $\tilde{\mathcal{M}}(c_1,c_2)$ are finite dimensional manifolds. Its local dimensions are*

$$\begin{aligned}
\dim_v\tilde{\mathcal{M}}^-(c_1) &= \mathrm{Ind}(c_1)-\mathrm{ind}_f(\mathrm{ev}_2(v))+m-1\\
\dim_v\tilde{\mathcal{M}}^+(c_2) &= \mathrm{ind}_f(\mathrm{ev}_1(v))+\dim_{\mathrm{ev}_1(v)}\mathrm{crit}(f)-\mathrm{Ind}(c_2)+m-1\\
\dim_v\tilde{\mathcal{M}}(c_1,c_2) &= \mathrm{Ind}(c_1)-\mathrm{Ind}(c_2)+m-1.
\end{aligned}$$

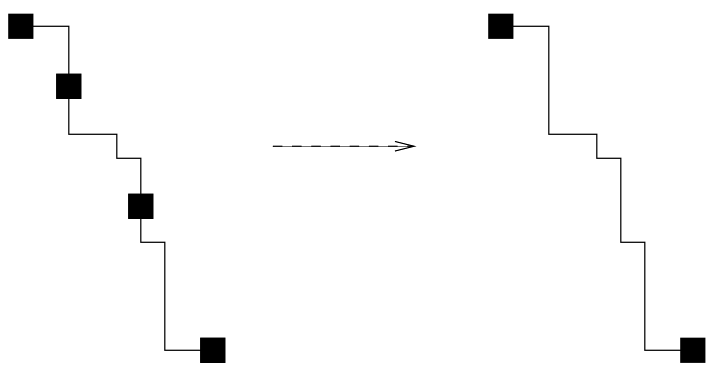
It follows from the Corollary above that for generic Riemannian metric g the moduli space of flow lines with cascades $\mathcal{M}(c_1, c_2)$ is a manifold of dimension $\mathrm{Ind}(c_1) - \mathrm{ind}(c_2) - 1$ in a neighbourhood of an element $v = ((x_j)_{1 \le j \le m}, (t_j)_{1 \le j \le m-1})$ all of whose $t_j > 0$. It remains to consider the case where some of the t_j's are zero. In a neighbourhood of such an element the number of cascades may vary and we need a gluing theorem to parametrise such a neighbourhood.

In view of the transversality, the subspace $\mathcal{B}(c_1, c_2)$ of $\mathcal{B}$ for $c_1, c_2 \in \mathrm{crit}(h)$ defined by

$$\mathcal{B}(c_1, c_2) := \{v \in \mathcal{B} : \mathrm{ev}_1(v) \in W_h^u(c_1),\ \mathrm{ev}_2(v) \in W_h^s(c_2)\}$$

is actually a submanifold. Recall that the evaluation map $\mathrm{EV}_n \colon \tilde{\mathcal{M}}^n \to \mathrm{crit}(f)^{2n}$ was defined by $\mathrm{EV}_n(v_1, \cdots, v_n) \mapsto (\mathrm{ev}_1(v_j)_{1 \le j \le n}, \mathrm{ev}_2(v_j)_{1 \le j \le n})$ and $A_n(c_1, c_2) \subset \mathrm{crit}(f)^{2n}$ was defined as the subspace of all tuples $((p_j)_{1 \le j \le n}, (q_j)_{1 \le j \le n}) \in \mathrm{crit}(f)^{2n}$ satisfying $p_1 \in W_h^u(c_1), q_n \in W_h^s(c_2)$, and $p_{j+1} = p_j$ for $1 \le j \le n-1$. We next define for T large enough the **pregluing map**

$$\#^0 : \mathrm{EV}_n^{-1}(A_n(c_1, c_2)) \times (T, \infty)^{n-1} \to \mathcal{B}(c_1, c_2).$$



The pregluing map

Let $\mathbf{v} = (v_k)_{1 \le k \le n} = ((x_{k,j})_{1 \le j \le m_k}, (t_{k,j})_{1 \le j \le m_k - 1}))_{1 \le k \le n} \in \mathrm{EV}_n^{-1}(A_n(c_1, c_2))$ and $\mathbf{R} = (R_j)_{1 \le j \le n-1} \in (T, \infty)^{n-1}$. For $1 \le p \le n$ and

$$\sum_{k=1}^{p-1} m_k - p + 2 \le i < \sum_{k=1}^{p} m_k - p + 1$$

set

$$s_i := t_{p, i - \sum_{k=1}^{p-1} m_k + p - 1}.$$

For

$$\sum_{k=1}^{p-1} m_k - p + 2 < i < \sum_{k=1}^{p} m_k - p + 1$$

put

$$y_i := x_{p,i-\sum_{k=1}^{p-1} m_k+p-1}.$$

Define in addition

$$y_1 := x_{1,1}, \quad y_{\sum_{k=1}^{n} m_k-n+1} := x_{n,m_n}.$$

Recall that $x_{p,m_p}(s)$ converges as s goes to ∞ to $\mathrm{ev}_2(v_p)$ and $x_{p+1,1}(s)$ converges as s goes to $-\infty$ to $\mathrm{ev}_2(v_p)$ as well. In particular, it follows that there exist $\xi_p(s), \eta_p(s) \in T_{\mathrm{ev}_2(v_p)}M$ such that $x_{p,m_p}(s) = \exp_{\mathrm{ev}_2(v_p)}(\xi_p(s))$ for large negative s and $x_{p+1,1}(s) = \exp_{\mathrm{ev}_2(v_p)}(\eta_p(s))$ for large positive s For $1 \leq p \leq n-1$ and

$$i = \sum_{k=1}^{p} m_k - p + 1$$

put

$$y_i := x_{p,m_p} \#^0_{R_p} x_{p+1,1}$$

where

$$x_{p,m_p} \#^0_{R_p} x_{p+1,1} :=$$

$$\begin{cases} x_{p,m_p}(s+R_p), & s \leq -R_p/2-1, \\ \exp_{\mathrm{ev}_2(v_p)}(\beta(-s-R_p/2)\eta_p(s+R_p)), & -R_p/2-1 \leq s \leq -R_p/2, \\ \mathrm{ev}_2(v_p), & -R_p/2 \leq s \leq R_p/2, \\ \exp_{\mathrm{ev}_2(v_p)}(\beta(s-R_p/2)\xi_p(s-R_p)), & R_p/2 \leq s \leq R_p/2+1, \\ x_{p+1,1}(s-R_p), & s \geq R_p/2+1. \end{cases}$$

Here $\beta : \mathbb{R} \to [0,1]$ is a cutoff function equal to 1 for $s \geq 1$ and equal to 0 for $s \leq 0$. We abbreviate for the number of cascades of the image of the pregluing map $\#^0$

$$\ell = \sum_{k=1}^{n} m_k - n + 1.$$

Then we define

$$v^0_{\mathbf{R}} := \#^0(\mathbf{v}, \mathbf{R}) := ((y_i)_{1 \leq i \leq \ell}, (s_i)_{1 \leq i \leq \ell-1}).$$

$v^0_{\mathbf{R}}$ will in general only lie "close" to $\tilde{\mathcal{M}}$. We will next construct $v_{\mathbf{R}} \in \tilde{\mathcal{M}}$ in a small neighbourhood of $v^0_{\mathbf{R}}$. It can be shown that there exists a Riemannian metric g on M such that $\mathrm{crit}(f)$, $W^u_h(c_1)$, and $W^s_h(c_2)$ are totally geodesic with respect to g. For $\zeta \in T_v\mathcal{B}(c_1, c_2)$ let

$$\rho(v, \zeta) : \mathcal{E}_v \to \mathcal{E}_{\exp_v(\zeta)}{}^9$$

be the parallel transport along the path $\tau \mapsto \exp_v(\tau\zeta)$ for $\tau \in [0,1]$. Define

$$F_v : T_v\mathcal{B}(c_1, c_2) \to \mathcal{E}_v, \quad \zeta \mapsto \rho(v,\zeta)^{-1}\mathcal{F}(\exp_v(\zeta)).$$

[9] We denote the restriction of $\mathcal{E}$ to $\mathcal{B}(c_1, c_2)$ also by $\mathcal{E}$.

Note that

$$DF_v(0) = D\mathcal{F}(v).$$

For large $R \in \mathbb{R}$ and $i \in \mathbb{N}_0$ define $\gamma^i_{\delta,R} : \mathbb{R} \to \mathbb{R}$ by

$$\gamma^i_{\delta,R}(s) := \begin{cases} (1-\beta[-s])\gamma_\delta(s+R) + \beta(s)\gamma_\delta(s-R) & i \geq 1 \\ (1-\beta[-s-R])\gamma_\delta(s+R) + \beta(s+R)\gamma_\delta(s-R) & i = 0 \end{cases}$$

Here β is a smooth cutoff function which equals 0 for $s \leq -1$ and equals 1 for $s \geq 0$. For a locally integrable real valued function $f \colon \mathbb{R} \to \mathbb{R}$ which has weak derivatives up to order k we define the norms

$$||f||_{k,p,\delta,R} := \begin{cases} ||\gamma^1_{\delta,R} f||_p & k = 0 \\ \sum_{i=0}^k ||\gamma^i_{\delta,R}\partial^i f||_p + ||f(0)|| & k > 0. \end{cases}$$

The $|| \, ||_{k,p,\delta,R}$-norm is equivalent to the $|| \, ||_{k,p,\delta}$-norm, but their ratio diverges as R goes to infinity. For $1 \leq i \leq \ell$ set

$$R^i := \begin{cases} R_q & i = \sum_{k=1}^q m_k - q + 1 \\ 0 & \text{else} \end{cases}$$

We modify the Banachspace norm (27) on $T_{v^0_{\mathbf{R}}}\mathcal{B}(c_1, c_2)$ in the following way. For $\zeta \in T_{v^0_{\mathbf{R}}}\mathcal{B}(c_1, c_2)$ we define

$$||\zeta||_{\mathbf{R}} := \sum_{j=1}^{\ell}(||\xi_{j,0}||_{1,p,\delta,R^j} + ||\xi_{j,1}|| + ||\xi_{j,2}||) + \sum_{j=1}^{\ell-1}|\delta_j|.$$

Analoguously, we define for $\eta \in \mathcal{E}_{v^0_{\mathbf{R}}}$ the norm

$$||\eta||_{\mathbf{R}} := \sum_{j=1}^{\ell} ||\eta_j||_{p,\delta,R^j}.$$

These norms were introduced in [FOOO, Section 18] and are required to guarantee the uniformity of the constant c in (34) below. Abbreviate

$$D_{\mathbf{R}} := DF_{v^0_{\mathbf{R}}}(0) = D_{v^0_{\mathbf{R}}}.$$

Recall that $R_j > T$ for $1 \leq j \leq n-1$. It is shown in [Sch1, Chapter 2.5] that if T is large enough then there exists a constant $c > 0$ independent of $\mathbf{R}$ and a right inverse $Q_{\mathbf{R}}$ of $D_{\mathbf{R}}$, i.e.

$$D_{\mathbf{R}} \circ Q_{\mathbf{R}} = \mathrm{id}, \tag{32}$$

such that

$$\mathrm{im} Q_{\mathbf{R}} = \ker(D_{\mathbf{R}})^{\perp} \tag{33}$$

and

$$||Q_{\mathbf{R}}\eta||_{\mathbf{R}} \leq c||\eta||_{\mathbf{R}}, \tag{34}$$

for every $\eta \in \mathcal{E}_{v^0_{\mathbf{R}}}$. Moreover, it follows from the construction of the pregluing map that there exist constants $c_0 > 0$ and $\kappa > 0$ such that

$$||F_{v^0_{\mathbf{R}}}(0)||_{\mathbf{R}} \leq c_0 e^{-\kappa||\mathbf{R}||}. \tag{35}$$

Now (32), (33), (34), (35) together with the Banach Fixpoint theorem imply that for $||\mathbf{R}||$ large enough there exists a constant $c > 0$ and a unique $\xi_{\mathbf{R}} := \xi_{\mathbf{v},\mathbf{R}} \in \ker(D_{\mathbf{R}})^{\perp}$ such that

$$F_{v^0_{\mathbf{R}}}(\xi_{\mathbf{R}}) = 0, \quad ||\xi_{\mathbf{R}}||_{\mathbf{R}} \leq c||F_{v^0_{\mathbf{R}}}(0)||. \tag{36}$$

For details see [Sch1, Chapter 2.5]. Let $U(\mathbf{v})$ be a small neighbourhood of $\mathbf{v}$ in $\mathrm{EV}_n^{-1}(A_n(c_1, c_2))$. Define the gluing map

$$\# : U(\mathbf{v}) \times (T, \infty)^{n-1} \to \tilde{\mathcal{M}}(c_1, c_2)$$

by

$$\#(\mathbf{w}, \mathbf{R}) := \exp_{w^0_{\mathbf{R}}}(\xi_{\mathbf{w},\mathbf{R}}).$$

Let

$$N := \sum_{k=1}^{n} m_k.$$

Then $\mathbb{R}^N$ acts on $U(\mathbf{v})$ by timeshift on each cascade of the first factor. The gluing map induces a map

$$\hat{\#} : (U(\mathbf{v})/\mathbb{R}^N) \times (T, \infty)^{n-1} \to \mathcal{M}(c_1, c_2)$$

which is an embedding for T large enough.

Proof of Theorem A.11: For $m \in \mathbb{N}$ let $\mathbb{N}_m := \{1, \ldots, m\}$. For $I \subset \mathbb{N}_{m-1}$ let

$$\mathcal{M}_{m,I}(c_1, c_2) \subset \mathcal{M}_m(c_1, c_2)$$

be the set of flow lines with cascades $((x_k)_{1 \leq k \leq m}, (t_k)_{1 \leq k \leq m-1})$ in $\mathcal{M}_m(c_1, c_2)$ such that

$$t_k > 0 \quad \text{if } k \in I, \qquad t_k = 0 \quad \text{if } k \notin I.$$

It follows from Theorem A.13 that for generic Riemannian metric g on M the space $\mathcal{M}_{m,I}(c_1, c_2)$ is a smooth manifold. Note that

$$\mathcal{M}_m(c_1, c_2) = \bigcup_{I \subset \mathbb{N}_{m-1}} \mathcal{M}_{m,I}(c_1, c_2)$$

and

$$\mathrm{int}\mathcal{M}_m(c_1, c_2) = \mathcal{M}_{m,\mathbb{N}_{m-1}}(c_1, c_2).$$

It follows from (28) in Theorem A.13 that

$$\dim \mathcal{M}_{m,\mathbb{N}_{m-1}}(c_1, c_2) = \mathrm{Ind}(c_1) - \mathrm{Ind}(c_2) - 1.$$

In particular, $\mathcal{M}_m(c_1, c_2)$ has for generic g the structure of a manifold with corners of dimension $\mathrm{Ind}(c_1) - \mathrm{Ind}(c_2) - 1$. We put for $n \in \mathbb{N}$

$$\mathcal{M}_{\leq n} := \bigcup_{1 \leq m \leq n} \mathcal{M}_m(c_1, c_2).$$

We show by induction on n that for generic g the set $\mathcal{M}_{\leq n}(c_1, c_2)$ can be endowed with the structure of a manifold of dimension $\mathrm{Ind}(c_1) - \mathrm{Ind}(c_2) - 1$. This is clear for $n = 1$. It follows from gluing that $\mathcal{M}_{\leq n}(c_1, c_2)$ can be compactified to a manifold with corners $\bar{\mathcal{M}}_{\leq n}(c_1, c_2)$ such that

$$\partial \bar{\mathcal{M}}_{\leq n}(c_1, c_2) = \bigcup_{I \subsetneq \mathbb{N}_n} \mathcal{M}_{n+1, I}(c_1, c_2) = \partial \mathcal{M}_{n+1}(c_1, c_2).$$

Hence

$$\mathcal{M}_{\leq n+1}(c_1, c_2) = \mathcal{M}_{\leq n}(c_1, c_2) \cup \mathcal{M}_{n+1}(c_1, c_2)$$

can be endowed with the structure of a manifold such that

$$\dim \mathcal{M}_{\leq n+1}(c_1, c_2) = \dim \mathcal{M}_{\leq n}(c_1, c_2) = \mathrm{Ind}(c_1) - \mathrm{Ind}(c_2) - 1.$$

This proves the theorem. $\square$

A.3 Morse-Bott homology

We assume in this subsection that M is a compact manifold.

Definition A.16 *We say that a quadruple (f, h, g, g_0) consisting of a Morse-Bott function f on M a Morse function h on* $\mathrm{crit}(f)$*, a Riemannian metric g on M and a Riemannian metric g_0 on* $\mathrm{crit}(f)$ *is a* **regular Morse-Bott quadruple** *if the following conditions hold.*

(i) *h and g_0 satisfy the Morse-Smale condition, i.e. stable and unstable manifolds of the gradient of h with respect to g_0 on* $\mathrm{crit}(f)$ *intersect transversally.*

(ii) *g is (f, h, g_0)-regular, in the sense of Definition A.14.*

Since the Morse-Smale condition is generic, see [Sch1, Chapter 2.3], it follows from Theorem A.13, that regular Morse-Bott quadruples exist in abundance. In particular, every pair (f, g) consisting of a Morse function f on M and a Riemannian metric g on M which satisfy the Morse-Smale condition gives a regular Morse-Bott quadruple.

For a pair (f, h) consisting of a Morse-Bott function f on M and a Morse-function h on $\mathrm{crit}(f)$, we define the chain complex $CM_*(M; f, h)$ as the $\mathbb{Z}_2$ vector space generated by the critical points of h which is graded by the index. More precisely, $CM_k(M; f, h)$ are formal sums of the form

$$\xi = \sum_{\substack{c \in \mathrm{crit}(h) \\ \mathrm{ind}(c) = k}} \xi_c c$$

with $\xi_c \in \mathbb{Z}_2$. For generic pairs (g, g_0) of a Riemannian metric g on M and a Riemannian metric g_0 on $\mathrm{crit}(f)$, the moduli-spaces of of flow lines with cascades $\mathcal{M}(c_1, c_2)$ is a smooth manifold of dimension

$$\dim \mathcal{M}(c_1, c_2) = \mathrm{ind}(c_1) - \mathrm{ind}(c_2) - 1.$$

If $\dim \mathcal{M}(c_1, c_2)$ equals 0, then $\mathcal{M}(c_1, c_2)$ is compact by Theorem A.10. Set

$$n(c_1, c_2) := \#\mathcal{M}(c_1, c_2) \bmod 2.$$

We define a boundary operator

$$\partial_k : CM_k(M; f, h) \to CM_{k-1}(M; f, h)$$

by linear extension of

$$\partial_k c = \sum_{\mathrm{ind}(c')=k-1} n(c, c')c'$$

for $c \in \mathrm{crit}(h)$ with $\mathrm{ind}(c) = k$. The usual gluing and compactness arguments imply that

$$\partial^2 = 0.$$

This gives rise to homology groups

$$HM_k(M; f, h, g, g_0) := \frac{\ker \partial_{k+1}}{\mathrm{im} \partial_k}.$$

Theorem A.17 *Let $(f^\alpha, h^\alpha, g^\alpha, g_0^\alpha)$ and $(f^\beta, h^\beta, g^\beta, g_0^\beta)$ be two regular quadrupels. Then the homologies $HM_*(M; f^\alpha, h^\alpha, g^\alpha, g_0^\alpha)$ and $HM_*(M; f^\beta, h^\beta, g^\beta, g_0^\beta)$ are naturally isomorphic.*

Proof: Pick some $\ell \in \mathbb{N}$ and choose for $1 \leq k \leq \ell$ smooth functions $f_k \in C^\infty(\mathbb{R} \times M, \mathbb{R}$ and smooth families of Riemannian metrics $g_{k,s}$ on TM with $f_k(s, \cdot)$ and $g_{k,s}$ independent of s for $|s| \geq T$ for some large enough constant $T > 0$ such that

$$f_1(-T) = f^\alpha, \quad f_\ell(T) = f^\beta, \quad f_k(T) = f_{k+1}(-T),\ 1 \leq k \leq \ell - 1$$

and

$$g_{1,-T} = g^\alpha, \quad g_{\ell,T} = g^\beta, \quad g_{k,T} = g_{k+1,-T},\ 1 \leq k \leq \ell - 1.$$

We assume further that $f_k(T)$ is Morse-Bott for $1 \leq k \leq \ell - 1$. For $2 \leq k \leq \ell$ let $r_k \in \mathbb{R}_\geq$ be nonnegative real numbers. Choose smooth Morse functions $h_1 \in C^\infty((-\infty, 0] \times \mathrm{crit}(f^\alpha), \mathbb{R})$, $h_{\ell+1} \in C^\infty([0, \infty) \times \mathrm{crit}(f^\beta), \mathbb{R})$, and $h_k \in C^\infty([0, r_k] \times \mathrm{crit}(f_{k+1}(T))$ and smooth families of Riemannian metrics $g_{0,1,s}$ on $\mathrm{crit}(f^\alpha)$ for $s \in (-\infty, 0]$ and $g_{0,\ell+1,s}$ on $\mathrm{crit}(f^\beta)$ for $s \in [0, \infty)$ and $g_{0,k,s}$ on $\mathrm{crit}(f_{k+1}(T))$ for $s \in [0, r_k]$. They are required to fulfill

$$h_1(s) = h^\alpha,\ g_{0,1,s} = g_0^\alpha,\ s \leq -T,$$

$$h_{\ell+1}(s) = h^\beta,\ g_{0,\ell+1,s} = g_0^\beta,\ s \geq T.$$

For $c_1 \in \mathrm{crit}(h^\alpha)$, $c_2 \in \mathrm{crit}(h^\beta)$, $m_1, m_2 \in \mathbb{N}_0$ we consider the following flow lines from c_1 to c_2 with $m = m_1 + m_2 + \ell$ cascades

$$(\mathbf{x}, \mathbf{T}) = ((x_k)_{1 \leq k \leq m}, (t_k)_{1 \leq k \leq m-1})$$

for $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_\geq := \{r \in \mathbb{R} : r \geq 0\}$ which satisfy the following conditions:

(i) x_k are nonconstant solutions of

$$\dot{x}_k(s) = -\nabla_{\tilde{g}_{k,s}} \tilde{f}_k(s, x_k),$$

where

$$\tilde{f}_k = \begin{cases} f^\alpha & 1 \leq k \leq m_1 \\ f_{k-m_1} & m_1 + 1 \leq k \leq m_1 + \ell \\ f^\beta & m_1 + \ell + 1 \leq k \leq m \end{cases}$$

and

$$\tilde{g}_k = \begin{cases} g^\alpha & 1 \leq k \leq m_1 \\ g_{k-m_1} & m_1 + 1 \leq k \leq m_1 + 1 + \ell \\ g^\beta & m_1 + \ell + 1 \leq k \leq m. \end{cases}$$

(ii) There exists $p_1 \in W^u_{h^\alpha}(c_1)$ and $p_2 \in W^s_{h^\beta}(c_2)$ such that $\lim_{s \to -\infty} x_1(s) = p_1$ and $\lim_{s \to \infty} x_m(s) = p_2$.

(iii) denote

$$\tilde{h}_k = \begin{cases} h^\alpha & 1 \leq k \leq m_1 - 1 \\ h_{k-m_1+1} & m_1 \leq k \leq m_1 + \ell - 1 \\ f^\beta & m_1 + \ell \leq k \leq m - 1 \end{cases}$$

and

$$\tilde{g}_{0,k} = \begin{cases} g_0^\alpha & 1 \leq k \leq m_1 - 1 \\ g_{0,k-m_1+1} & m_1 \leq k \leq m_1 + 1 + \ell - 1 \\ g_0^\beta & m_1 + \ell \leq k \leq m - 1. \end{cases}$$

For $1 \leq k \leq m - 1$ there are Morse-flow lines y_k of h, i.e. solutions of

$$\dot{y}_k(s) = -\nabla_{\tilde{h}_{0,k,s}} \tilde{h}_k(s, y_k),$$

such that

$$\lim_{s \to \infty} x_k(s) = y(0)$$

and

$$\lim_{s \to -\infty} x_{k+1}(s) = y(t_k).$$

(iv) $t_{k+m_1-1} = r_k$ for $2 \leq k \leq \ell$.

For generic choice of the data the space of solutions of (i) to (iv) is a smooth manifold whose dimension is given by the difference of the indices of c_1 and c_2. If $I(c_1) = I(c_2)$ then this manifold is compact and we denote by $n(c_1, c_2) \in \mathbb{Z}_2$ its cardinality modulo 2. We define a map

$$\Phi^{\alpha\beta} : CM_*(M; f^\alpha, h^\alpha) \to CM_*(M; f^\beta, h^\beta)$$

by linear extension of

$$\Phi^{\alpha\beta} c = \sum_{\substack{c' \in \mathrm{crit}(h^\beta) \\ \mathrm{ind}(c') = \mathrm{ind}(c)}} n(c, c') c'$$

where $c \in \mathrm{crit}(f^\alpha)$. Standard arguments, see [Sch1, Chapter 4.1.3], show that $\Phi^{\alpha\beta}$ induces isomorphisms on homologies

$$\phi^{\alpha\beta} : HM_*(M; f^\alpha, h^\alpha, g^\alpha, g_0^\alpha) \to HM_*(M; f^\beta, h^\beta, g^\beta, g_0^\beta)$$

which satisfy

$$\phi^{\alpha\beta} \circ \phi^{\beta\gamma} = \phi^{\alpha\gamma}, \quad \phi^{\alpha\alpha} = \mathrm{id}.$$

This proves the theorem. $\square$

Theorem A.17 allows us to define the **Morse-Bott homology** of M by

$$HM_*(M) := HM_*(M; f, h, g, g_0)$$

for some regular quadruple (f, h, g, g_0). Either for the special case of a Morse function or the case where f vanishes identically one obtains that Morse-Bott homology is isomorphic to Morse homology. Hence we have proved the following Corollary.

Corollary A.18 *Morse-Bott homology of a compact manifold M is isomorphic to the Morse-homology of M and hence also to the singular homology of M.*

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